Second Order Calculation of the Correlation Function for a Four Quark State

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Abstract

The large number of scalar meson states below 2 GeV contradicts the expected number derived from a quark-antiquark description. One possibility is that one or more of the light scalar mesons can be described as four quark states composed of quark-antiquark pairs ($q\bar{q}q\bar{q}$). This scenario has been explored with sum-rule methods in Quantum Chromodynamics (QCD) to leading-order in the strong coupling constant. Higher loop contributions are significant in the QCD sum-rule analysis of quark-antiquark scalar states and a similar situation could occur in the four-quark case. In this thesis the leading order and pieces of the second order terms of the correlation function, as needed to study properties of a four-quark state via a QCD sum-rule, are calculated in the chiral limit (*i.e.* massless quarks) in QCD. Operator mixing related to renormalization of the composite operators appearing in the correlation function first contributes at second order. The result for the second order contributions to the correlation function indicate that operator mixing must be addressed before using proper dispersion relations to link this calculation with the mass of an existing state.

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In loving memory of Henri and Fernande Mercier

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QFT:	Quantum Field Theory
QED:	Quantum Electrodynamics
QCD:	Quantum Chromodynamics
PI:	Path Integral
MS:	Minimal Subtraction

Chapter 1

Introduction

Gaining a better understanding of the fundamental constituents of matter has long been a goal of physicists. Furthermore, developing an understanding of how these fundamental particles interact with one another is crucial to build upon our knowledge of the world around us. Any theory of how the universe works can not be considered complete until it can describe how things interact at the most basic, elementary level. Physicists are aware of the four fundamental forces acting over a wide variety of ranges: gravitation, electromagnetic, strong and weak. Several well-tested theories have been developed in the past century to describe these four forces. For gravity we have general relativity, for the electromagnetic force we have quantum electrodynamics (QED) [1], the weak and electromagnetic forces have been unified by the electroweak theory (Weinberg-Salam-Glashow) [2] and for the strong force we have quantum chromodynamics (QCD) [3]. Throughout history many leaps have been made towards understanding these forces, and through the use of better and more accurate technologies, we have been able to probe regions of space ranging from the extremely small to the extraordinarily large. In the process we have discovered a vast array of particles that make up the matter in the world around us.

The ongoing search for the smallest constituents of matter has been a long and arduous process. Rutherford's discovery that atoms were made up of compact nuclei gave physicists a model of the inner structure of atoms. When the neutron was discovered in 1932 by James Chadwick, it was believed that the proton, neutron and electron were the fundamental constituents of all matter. However, further experimentation in particle accelerators at higher energies would reveal a host of other particles which could be considered equally as fundamental. These particles have been classified into the following groups:

- <u>Leptons</u>- are spin 1/2 particles (Fermions) that are sensitive only to the weak and electromagnetic interactions and whose lepton number is conserved except for some cases in which this is violated due to neutrino oscillation [4]. Leptons specifically include the electron, muon and tau (e^-, μ^-, τ^-) along with their respective neutrinos and their associated antiparticles.

- <u>Hadrons</u>- are sensitive to the strong force as well as the weak and electromagnetic interactions. It is important to note that while hadrons were once thought of as elementary particles, they are now considered bound states of even more fundamental particles (quarks). They are split

Property	Quark Type					
	d	u	s	c	b	t
Q - Electric Charge	$-\frac{1}{3}e$	$+\frac{2}{3}e$	$-\frac{1}{3}e$	$+\frac{2}{3}e$	$-\frac{1}{3}e$	$+\frac{2}{3}e$
I - Isospin	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
I_z - Isospin Projection	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	0	0
S - Strangeness	0	0	-1	0	0	0
C - Charm	0	0	0	$+1$	0	0
B - Bottomness	0	0	0	0	-1	0
T - Topness	0	0	0	0	0	$+1$
M - Mass $\left(\frac{GeV}{c^2}\right)$	≈ 0.01	≈ 0.01	≈ 0.2	1.3	4.2	174

Table 1.1: Table of Quark "Flavours" and their Properties

into two categories:

  – Baryons- Fermions whose baryon number appears to be conserved, e.g. (p, n, Λ^0, Σ^+, Σ^-, Σ^0,...)

  – Mesons - Bosons (integer spin) whose meson number is not conserved, e.g. (π^+, π^-, π^0, K^+, K^-, K^0,...)

- Gauge Bosons- are the particles that mediate the fundamental interactions and they are not conserved in number, e.g. (γ, Z^0, W^+, W^-, gluons)

Although taking an inventory of the known particles has been useful, it is perhaps more important to physicists that we understand the inner workings of these fundamental constituents of matter. That way we will be able to apply a mathematical framework to them in order to make predictions about their interactions, how they decay and the possibility of finding new particles. When examining the makeup of particles at such small distances, we must have a theory that can incorporate the strong force.

Quantum-chromodynamics (QCD), justified by a great deal of experimental and theoretical evidence (see [5] for an overview), has led physicists to the understanding that hadrons, once thought to be elementary, are best understood as bound states of fundamental particles known as quarks. In particular baryons are usually interpreted as a bound state of three quarks, qqq (or antiquarks $\bar{q}\bar{q}\bar{q}$), while mesons are typically understood as a quark-antiquark pair, $q\bar{q}$. Quarks are fermions (particles with spin $\frac{1}{2}$) that interact via the strong force and have positive parity and an additive baryon number ($\mathcal{B}$) of $\frac{1}{3}$. Table 1.1 lists the various additive quantum numbers (or "flavours") for the different types of quarks [up (u), down (d), strange (s), charm (c), bottom (b) and top (t)].

Meson	Type of Multiplet	Number of States
$a_0(1450)$	isotriplet	3
$K_0(1430)$	isodoublet	2
$\bar{K}_0(1430)$	isodoublet	2
$f_0(1370)$	isosinglet	1
$f_0(1500)$	isosinglet	1
$f_0(1710)$	isosinglet	1
Total		10

Table 1.2: Some Light Scalar Mesons

While these new quantum numbers or "flavours" may seem to have been assigned after the fact, these assignments were actually made after the phenomenological observation that the electric charge of a hadron (Q) can be related to its other quantum numbers via the Gell-Mann-Nishijima formula [6]

$$Q = I_z + \frac{\mathcal{B} + S + C + B + T}{2} = I_3 + \frac{\mathcal{Y}}{2}\,, \tag{1.1}$$

where $\mathcal{Y}$ is known as the strong hypercharge. As we will see later on, it is important to note that the up, down and strange quarks are essentially massless at the energy scales used in this thesis.

A unique feature of QCD is that we can classify and understand many characteristics of hadrons by introducing an extra degree of freedom through the addition of a new quantum number: "colour" [7]. Originally this extra degree of freedom was introduced to deal with Pauli exclusion principle problems arising from the analysis of the Δ^{++} baryon. The strong force is now understood as the manifestation of quark colour interactions. It is also interesting to note that the bound states of quarks are "colourless" (*i.e.* there is no net colour). The gauge bosons that mediate the strong colour charge through which the quarks interact are called gluons.

One category of hadrons in particular garners some special attention and is the primary motivation for the caculation performed in this thesis. The scalar mesons are mesons with a total spin of 0 and even parity ($J^P = 0^+$). As noted by the Particle Data Group "The identification of the scalar mesons is a long standing puzzle" [8]. A mathematical analysis of the scalar meson states would have us believe that the light scalar mesons should be grouped into a singlet and an octet. However this is not the case. As indicated in Table 1.2, there are in fact too many scalar mesons below 2 GeV to be explained by a quark-antiquark $q\bar{q}$ model.

Since there are too many scalar mesons that can not be explained in a simple $q\bar{q}$ model it is important to examine the other possible configurations that could lead to these states. One possibility is that these states actually include glueballs. Although coloured, gluons can combine to form colourless bound states. Glueballs are states composed entirely of gluons with no valence quarks who interact via the strong force. These states have been extremely difficult to detect since they appear to be mixed with ordinary meson states. Another possible explanation, which is examined in this thesis, is that these scalar mesons actually include exotic multiquark states, specifically four-quark states composed of quark-antiquark pairs ($q\bar{q}q\bar{q}$) as first conjectured by Gell-Mann [9].

It is interesting to note that multiquark states are not forbidden by colour confinement and are theoretically possible in QCD. It is also possible to obtain hadron-hadron potentials from four-quark configurations [10] . Experiments have given us several potential four-quark candidates [8, 11, 12] ($a_0(980), f_0(980), f_1(1420), f_2(1565)$ and $\Psi(4040)$). The manner in which these particles decay suggests their connection to a multiquark state. For example, the dominant decay of the $a_0(980)$ particle is to a η meson and a π meson. Similarly, the $f_0(980)$ decays into a π-π pair. Since a common trend among the aforementioned scalar mesons is to decay into $q\bar{q}$ pairs, there is further motivation to suggest that these scalar mesons may in fact be $q\bar{q}q\bar{q}$ states.

"Four-quark systems have been analyzed using several different models. Whether or not the existence of the four-quark state can be confirmed depends very much on the model being used and not many models have a base in QCD" [13]. For my research I have calculated the correlation function used by QCD sum rules to analyze a scalar four-quark system composed of two quark-antiquark ($q\bar{q}$) pairs. In order to determine the mass of a four-quark state we must calculate the operator product expansion for the correlation function $\Pi(q^2)$. In my case $\Pi(q^2)$ can be determined from the expression

$$\Pi(q^2) = i \int d^4x e^{iq\cdot x} \langle 0|T(J(x)J(0))|0\rangle \,, \tag{1.2}$$

where $|0\rangle$ represents the vacuum state, and the scalar composite operator $J(x)$ is an appropriate scalar current corresponding to the $q\bar{q}q\bar{q}$ state, the choice of which, concerns a portion of my thesis. The $T()$ is a time-ordering operator that aligns the operators in the brackets with the latest to the left. The Feynman diagram analysis of the matrix element $\langle 0|T(J(x)J(0))|0\rangle$ has a simple loop expansion in α_s (strong coupling constant) analogous to a perturbative expansion in α with the added exception that this expansion is also done in higher orders of $\hbar$. The analysis of this four-quark state has been done when only the leading order term for the above expansion has been considered for a particular scalar current operator $J(x)$ [11, 12, 13, 14, 15]. The goal of my thesis is the calculation of the next order corrections (terms proportional to α_s) to $\Pi(q^2)$ to facilitate future improvements in the accuracy of [11, 12, 13, 14, 15]. Indeed, it is known that for $q\bar{q}$ scalar mesons and scalar glueballs, higher-order terms in α_s are significant in QCD sum rules [16] The correlation function $\Pi(q^2)$ has been calculated analytically to order α_s using dimensional regularization, in a form that will be ready for renormalization using the MS-scheme (minimal subtraction). Since the motivation for analyzing a four-quark state comes from discrepancies in the light scalar mesons we can assume that our quark masses will be small compared to q^2. This provides the justification for working in the chiral limit, and we take the quark masses to be zero. It is also important to take note that additional complications will arise from the renormalization of the composite operator in the operator product expansion, which is an extremely complicated process and is beyond the scope of this thesis.

In Chapter 2 the basic elements of quantum field theory that are required for this calculation are explained. In particular, Sections 2.3 and 2.4 on Wick's theorem and Gauge theories respectively, deal with concepts that are vital to the progression of the calculation. Chapter 3 deals with the theory behind QCD, detailing the background of elements necessary to this calculation such as Feynman diagrams, interpolating fields (*i.e.* currents), the correlation function, renormalization, dimensional regularization and Feynman integrals. A derivation of the fundamental Feynman integral as well as the derivation for a specific three-loop Feynman integral necessary for the calculation of the correlation function are performed in Sections 3.9 and 3.10 respectively.

Chapter 4 contains my calculation of the correlation function for the four-quark state to leading order and next-to-leading order. The various terms are categorized and calculated individually. By creating a link between the different categories of terms involved in this calculation and their corresponding Feynman diagrams, it is easier to visualize this process. Chapter 4 begins with the separation of the correlation function into lowest order and next highest order terms. Section 4.1 contains the calculation of the lowest order terms, while Sections 4.2 through 4.6 deal with the calculation of the next highest order pieces. Chapter 5 presents and discusses the result for the correlation function $\Pi(q^2)$. The Appendix contains the notation and conventions used in this thesis.

Chapter 2

Quantum Field Theory Review

Quantum field theory (QFT) for fundamental interactions combines quantization, classical field theory and special relativity to study processes involving many-particle systems, including cases where the number of particles change. QFT is the heart of modern elementary particle physics, while providing tools and insights into other areas of physics such as atomic physics, nuclear physics, condensed matter physics and astrophysics. Quantum electrodynamics (QED), the field theory that describes the interaction of electrically charged particles with each other by the exchange of photons, is considered by many as the "best fundamental theory we have" [17] due to its extremely accurate predictions of things like the anomalous magnetic moment of the muon [18]. Quantum chromodynamics (QCD) is the relativistic field theory that studies quark interactions with the strong force carriers (gluons) and will be the theory underlying the content of this thesis.

2.1 Canonical Quantization

This section illustrates the basic concepts for the canonical quantization of classical real scalar fields. The ideas in this section can be extended to field theories that contain spin $\frac{1}{2}$ states and gauge fields. We begin with a classical field theory described by a Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ where the field satisfies the Euler-Lagrange equation

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} - \frac{\delta \mathcal{L}}{\delta \phi} = 0\,. \tag{2.1}$$

We wish to "quantize" this theory by reinterpreting the fields as operators that obey appropriate commutation relations. We will consider ϕ and its conjugate momentum operator π

$$\pi(\mathbf{x}) \equiv \frac{\delta \mathcal{L}}{\delta \dot{\phi}(\mathbf{x})}\,, \tag{2.2}$$

as operators that satisfy the following equal time commutation relations:

$$\begin{aligned} [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \\ [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = [\pi(\mathbf{x}, t)&, \pi(\mathbf{y}, t)] = 0\,. \end{aligned} \tag{2.3}$$

The Hamiltonian

$$H = \int d^3x [\pi(x)\partial_0\phi(x) - \mathcal{L}(x)] \tag{2.4}$$

controls the dynamics of the system and results in the Heisenberg equations of motion

$$\partial_0\phi(\mathbf{x},t) = i[H,\phi(\mathbf{x},t)] \tag{2.5}$$

$$\partial_0\pi(\mathbf{x},t) = i[H,\pi(\mathbf{x},t)]\,. \tag{2.6}$$

We can cast the above formulation into an appropriate free theory piece and an interaction piece and then solve the theory perturbatively. Taking this approach will eventually lead to the derivation of the Feynman rules for that theory.

2.2 Path Integral Formalism

It is important to realize that there are different ways to quantize a relativistic field theory each with their own advantages and disadvantages. While the canonical formulation presents the underlying concepts of quantum field theory in a more accessible manner it lacks the computational power of the path integral formalism. Nonetheless both formulations lead to the same set of Feynman rules describing interactions in the QFT. While, for the purposes of this thesis, we will not dig too deep into the Path Integral formalism it is nonetheless important to make mention of its usefulness to quantum field theory specifically in its suitability for handling non-abelian gauge theories such as QCD [19].

2.3 Wick's Theorem

When performing calculations in a quantum field theory involving scalar fields we will need to evaluate expressions of the form

$$\langle 0|T(\phi(x_1)\phi(x_2)...\phi(x_n))|0\rangle\,. \tag{2.7}$$

Wick's theorem [20] is a very useful way to simplify calculations of this sort. Within Wick's theorem, we separate $\phi(x)$ into

$$\phi(x) = \phi^+(x) + \phi^-(x) \tag{2.8}$$

where $\phi^+(x)$ is proportional to the annihilation operator $a_{\mathbf{p}}$ and $\phi^-(x)$ is proportional to the raising operator $a^\dagger_{\mathbf{p}}$. We also define the "normal ordering" operation $N()$ to place operators in normal order (*i.e.* all $a_{\mathbf{p}}$'s are to the right of all $a^\dagger_{\mathbf{p}}$'s). For example,

$$N(a_{\mathbf{p}}a^\dagger_{\mathbf{k}}a_{\mathbf{q}}a^\dagger_{\mathbf{l}}) = a^\dagger_{\mathbf{k}}a^\dagger_{\mathbf{l}}a_{\mathbf{p}}a_{\mathbf{q}}\,. \tag{2.9}$$

We must also define the contraction of two fields as

$$\overbrace{\phi(x)\phi(y)} \equiv \langle 0|T(\phi(x)\phi(y))|0\rangle = \begin{cases} \langle 0|\phi(x)\phi(y)|0\rangle & \text{for } x^0 > y^0 \\ \langle 0|\phi(y)\phi(x)|0\rangle & \text{for } y^0 > x^0 \end{cases}$$
$$= \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for } x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & \text{for } y^0 > x^0\,, \end{cases} \tag{2.10}$$

which is the Feynman propagator $\overbrace{\phi(x)\phi(y)} = \langle 0|T(\phi(x)\phi(y))|0\rangle = D_F(x-y)$. Wick's Theorem states that

$$T(\phi(x_1)\phi(x_2)...\phi(x_n)) = N\{\phi(x_1)\phi(x_2)...\phi(x_n) + \text{all possible contractions of the fields}\}\,, \tag{2.11}$$

which means that we can break down the matrix elements of the time-ordered products of many fields into two point functions that are easier to manage. Wick's theorem can also be expressed in an alternate notation [20]

$$\begin{aligned} T(\phi(x_1)\cdots\phi(x_n)) =& N(\phi(x_1)\cdots\phi(x_n)) \\ &+ \sum_{k<l} N(\phi(x_1)\cdots\phi(\hat{x}_k)\cdots\phi(\hat{x}_l)\cdots\phi(x_n))\langle 0|T(\phi(x_k)\phi(x_l))|0\rangle + \cdots \\ &+ \sum_{k_1<k_2<\cdots<k_{2p}} N(\phi(x_1)\cdots\phi(\hat{x}_{k_1})\cdots\phi(\hat{x}_{k_{2p}})\cdots\phi(x_n)) \\ &\times \sum_P \langle 0|T(\phi(x_{k_{P_1}})\phi(x_{k_{P_2}}))|0\rangle\cdots\langle 0|T(\phi(x_{k_{P_{2p-1}}})\phi(x_{k_{P_{2p}}}))|0\rangle \\ &+ \text{all possible combinations and permutations}\,. \end{aligned} \tag{2.12}$$

In this form the hat above a term indicates that this term is to be removed from the product and the sum $\sum_P$ takes into account all possible permutations that lead to different expressions. Wick's theorem can be extended to fermions by taking into account minus signs resulting from the anti-commutator algebra. It is also important to note that while there are non-perturbative effects that would cause terms of the form $\langle 0|N(\text{any operator})|0\rangle$ to be non-zero [21], in the strict perturbative expansion these terms are ignored, leaving only the final line of (2.12). For our purposes, we will assume a strict perturbative expansion and consider that the normal order product, $N()$, will cause any terms with uncontracted operators to equal zero.

2.4 Gauge Theories and Gauge Fixing

Gauge theories are a class of physical theories in which symmetry transformations can be performed both locally and globally. Global symmetry implies that the physical observables are invariant under an identical transformation carried out at every space-time point. The local symmetry requirement implies that we should be able to perform these transformations in one area of space-time without altering the physics in another region of space-time.

When performing calculations in a given field theory the field variables that are utilized are not usually the physical quantities we wish to calculate. The physical quantities that we wish to calculate are considered to be equivalence classes of gauge fields. The concept of Gauge fixing is used in order to reduce the redundant degrees of freedom inherent in the field variables that we use in calculations in QCD. By choosing an appropriate gauge we can reduce the complexity of our calculation dramatically. It is important to note that by using gauge invariant operators in our current $J(x)$ we will be able to guarantee that our correlation function will also be gauge invariant [22, 23]. For the purposes of this thesis we will work in the Feynman gauge (gauge parameter $a = 1$).

Chapter 3

QCD (Quantum Chromodynamics)

QCD is a field theory that explains the strong interactions between quarks and gluons. In QCD we represent quarks as spin 1/2 Dirac spinor fields with fractional electric charge and the gluons as massless spin 1 gauge fields which interact with the quarks and themselves.

We define q^A_α to be the quark fields where $A = 1, 2, ..., N_f$ is the flavour index corresponding to $q^1_\alpha = u =$ up, $q^2_\alpha = d =$ down, $q^3_\alpha = s =$ strange, *etc*... The subscript $\alpha = 1, 2, ...N$ refers to the colour of the quark, for QCD $N = 3$ (*e.g.* 1=red, 2=blue, 3=green). We can write the Lagrangian density for free, massless quarks as

$$\mathcal{L}_0(x) = \frac{i}{2}\bar{q}^A_\alpha(x)\gamma^\mu\partial_\mu q^A_\alpha(x) - \frac{i}{2}\left[\partial_\mu\bar{q}^A_\alpha(x)\right]\gamma^\mu q^A_\alpha(x)\,, \tag{3.1}$$

where all the sums over repeated indices are implicit. While this Lagrangian density is invariant under global gauge colour transformations our needs require a Lagrangian that is invariant under local gauge transformations

$$q^A_\alpha(x) \longrightarrow q'^A_\alpha(x) \equiv \left[e^{-igT_a\theta_a(x)}\right]_{\alpha\beta} q^A_\beta(x)\,, \tag{3.2}$$

where $\theta_z(x)$ are real space-time functions, g is a real dimensionless coupling constant and the T_a are the generators in the fundamental representation of SU(N). We also need to substitute the usual derivative ∂^μ by a covariant derivative

$$\delta_{\alpha\beta}\partial^\mu \longrightarrow D^\mu_{\alpha\beta} \equiv \delta_{\alpha\beta}\partial^\mu - igT^a_{\alpha\beta}B^\mu_a(x)\,, \tag{3.3}$$

where $B^\mu_a(x)$ are the gluon fields in the adjoint representation of $SU(N)$. So for the transformations

$$\begin{aligned} q^A_\alpha(x) &\longrightarrow q'^A_\alpha(x) = q^A_\alpha(x) - igT^a_{\alpha\beta}\delta\theta_a(x)q^A_\beta(x) \\ B^\mu_a(x) &\longrightarrow B'^\mu_a(x) = B^\mu_a(x) + gf_{abc}\delta\theta_b(x)B^\mu_c(x) - \partial^\mu\delta\theta_a(x) \end{aligned} \tag{3.4}$$

the $\delta\theta_a(x)$ are infinitesimal functions that characterize the transformation and f_{abc} are the structure constants of $SU(N)$. The Lagrangian density that is invariant under local gauge transformations and describes free massless quark fields and their interaction with the gluon fields is

$$\begin{aligned} \mathcal{L}_0 &= \frac{i}{2}\bar{q}^A_\alpha(x)\gamma_\mu D^\mu_{\alpha\beta}q^A_\alpha(x) - \frac{i}{2}[D^{\mu *}{}_{\beta\alpha}\bar{q}^A_\alpha(x)]\gamma_\mu q^A_\alpha(x) \\ &= \frac{i}{2}\bar{q}^A_\alpha(x)\gamma_\mu\partial^\mu q^A_\alpha(x) - \frac{i}{2}[\partial^\mu\bar{q}^A_\alpha(x)]\gamma_\mu q^A_\alpha(x) + \frac{1}{2}g\bar{q}^A_\alpha(x)\lambda^a_{\alpha\beta}\gamma_\mu q^A_\beta(x)B^\mu_a(x)\,, \end{aligned} \tag{3.5}$$

where $T^a = \frac{\lambda^a}{2}$.

If we take into account the gluon dynamics we will require another term in the Lagrangian. First let us clean up our notation

$$B^\mu(x) \equiv igT_a B_a^\mu(x), \quad D^\mu \equiv \mathbf{I}\partial^\mu - B^\mu(x) \,. \tag{3.6}$$

Then we can define the antisymmetric field strength tensor $F^{\mu\nu}$ as

$$F^{\mu\nu} \equiv -[D^\mu, D^\nu] = \partial^\mu B^\nu(x) - \partial^\nu B^\mu(x) - [B^\mu(x), B^\nu(x)] \,. \tag{3.7}$$

The quantity

$$Tr[F^{\mu\nu}(x)F_{\mu\nu}(x)] = -\frac{g^2}{2}F_a^{\mu\nu}(x)F_{\mu\nu}^a(x) \tag{3.8}$$

transforms as a scalar and is invariant under local gauge transformations. Adding a term proportional to the above gives us the following Lagrangian density:

$$\mathcal{L}_0 = \frac{1}{2g^2}Tr[F^{\mu\nu}(x)F^{\mu\nu}(x)] + i\bar{q}^A(x)\gamma_\mu D^\mu q^A(x) \,, \tag{3.9}$$

which accounts for the gluon dynamics in addition to kinetic terms for massless quarks. The gluons must be massless to preserve gauge invariance. Finally we must account for the masses of the quarks in the theory. By adding the gauge-invariant term containing the quark mass m_A to the above Lagrangian density we obtain

$$\mathcal{L}_0 = \frac{1}{2g^2}Tr[F^{\mu\nu}(x)F_{\mu\nu}(x)] + i\bar{q}^A(x)\gamma_\mu D^\mu q^A(x) - m_A\bar{q}^A(x)q^A(x) \,, \tag{3.10}$$

which is the Lagrangian for classical chromodynamics. Built into this Lagrangian is a kinetic term for massless gluon fields, kinetic terms for quark fields of different masses, interaction of the quark fields with the gluons and the self interaction of the gluons. While we have built the above Lagrangian to be $SU(N)$ invariant it is important to note that in nature we observe the specific case of an $SU(3)$ symmetry.

3.1 Feynman Diagrams in QCD

Feynman diagrams play an important role in the conceptual understanding of the processes in a quantum field theory. When performing calculations in our theory we are unable to find exact expressions for probability amplitudes for processes to occur. It becomes necessary to expand these amplitudes as a perturbation series in the strength of the coupling constant of the theory. The method of Feynman diagrams provide a relatively simple and aesthetically pleasing way of visualizing this expansion. For the purposes of this thesis we need only look at the diagrams representing quarks, gluons and their interaction.

The gluon propagator shown in Figure 3.1 is

$$iD^{(0)\mu\nu}_{ab}(k) \equiv \int d^4x e^{ik\cdot x}\langle 0|T(A^\mu_a(x)A^\nu_b(0))|0\rangle = \delta_{ab} i\left(-\frac{g^{\mu\nu}}{k^2} + (1-a)\frac{k^\mu k^\nu}{k^4}\right)e^{-ik\cdot(x-y)}\,, \tag{3.11}$$

where a is the gauge parameter. For our purposes we will use the Feynman/Fermi gauge ($a = 1$).

The quark propagator shown in Figure 3.2 is

$$iS^{AB}_{\alpha\beta}(\not k) \equiv \int d^4x e^{ik\cdot x}\langle 0|T(q^A_\alpha(x)\bar{q}^B_\beta(0))|0\rangle = i\frac{\not k + m}{k^2 - m^2}\,. \tag{3.12}$$

When calculating the quark-gluon interaction, given in Figure 3.3, we must calculate the following quantity:

$$\langle 0|T(\psi(x)\bar{\psi}(0)A^\mu_a(y)\exp\left[i\int \mathcal{L}_{int}(w)d^4w\right])|0\rangle\,. \tag{3.13}$$

After expanding the exponential containing the interaction term of the Lagrangian, taking the first non-zero term from this expansion and contracting the fields, we obtain the following result for the quark gluon interaction:

$$\begin{aligned}&\langle 0|T(\psi(x)\bar{\psi}(0)A^\mu(y)\left(-i\frac{g}{2}\lambda\right)\bar{\psi}(w)A^\nu(w)\gamma^\nu\psi(w))|0\rangle\\ =&iS(x-w)iS(w)iD^{\mu\nu}_{ab}(y-w)i\frac{g}{2}\lambda^b\gamma^\nu\,.\end{aligned} \tag{3.14}$$

Note that the quark gluon interaction contains the quark propagators $S(x-w)$ and $S(w)$ as well as the gluon propagator $D^{\mu\nu}_{ab}(y-w)$ corresponding to the quark and gluons fields respectively. It is understood that the $-i\frac{g}{2}\lambda$ term corresponds to the vertex in Figure 3.3 and the propagators are taken to be implicit when calculating the quark-gluon interaction.

Figure 3.1: Gluon propagator

Figure 3.2: Quark propagator

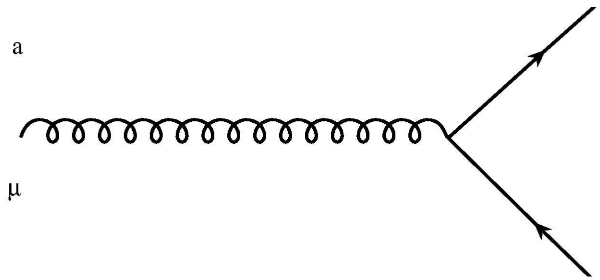

Figure 3.3: Quark-gluon interaction

3.2 SU(3) and the Naive Quark Model

While a study of the applications and details of group theory in quantum field theory is beyond the scope of this paper it is important to point out some of the general results from such analysis. Gell-Mann first noticed that when plotting hadrons with the same spin and parity (J^P) according to Isospin T_3 and hypercharge $Y = B + S$ ($B \equiv$ baryon number and $S \equiv$ strangeness) distinct patterns appeared [24].

These patterns corresponded to the eight dimensional and ten dimensional representations of the group SU(3), which is the group of 3×3 unitary matrices with determinant equal to 1. Gell-Mann believed that hadrons could be understood as bound states having been formed by fundamental representations of SU(3). These states were called quarks and labeled as u (up), d (down) and s (strange) with electric charges $q_u = 2/3$, $q_d = -1/3$, $q_s = -1/3$. If the masses of the quarks are approximately degenerate (which the lightest three are) then the QCD Lagrangian density, and consequently the dynamics of the theory, will be invariant under SU(3) flavour transformations. Later on it was discovered that in order to account for the spin of the Δ^{++} another quantum number, colour, needed to be incorporated into the theory.

Understanding what combinations of quarks will lead to the creation of a legitimate elementary particle is important not only for testing the quark model but also to finding new and exotic particles. Because in nature quarks always seem to form bound states that are colourless, we would require that any quark combination would lead to a group decomposition that contains a singlet. As of now the only particles in nature that have been definitively observed have been quark-antiquark ($q\bar{q}$) combinations (mesons) $3 \otimes 3^* = 1 \oplus 8$ and three-quark (qqq) combinations $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$ which both have SU(3) colour singlets.

It is the focus of my thesis to calculate the correlation function $\Pi(q^2)$, to order α_s, of a quark-antiquark pair ($q\bar{q}q\bar{q}$) particle $3 \otimes 3^* \otimes 3 \otimes 3^* = 1 \oplus \cdots$ which also has a colour singlet representation and as such should be allowed to exist by the quark model.

3.3 Current (Interpolating Fields)

In order to connect the vacuum to a hadronic state with specific quantum numbers we must make sure that the operator we use has the same quantum numbers as the state, *i.e.*

$$\langle \text{Hadronic State}|J|0\rangle \neq 0\,. \tag{3.15}$$

Depending on which states we desire to examine, we must make sure to choose these operators accordingly. We define the current $J(x)$ to be a local composite operator represented by quark fields with the appropriate quantum numbers to fit our hadronic state. We also call $J(x)$ an interpolating field for a particular hadronic state.

A useful example is the ρ-meson. It is a vector particle ($J=1$), has even parity ($P=+1$) and is an isotriplet ($I=+1$) and so we construct a current $J(x)$

$$J^{\mu}_{\rho}(x) = \frac{1}{2}\left[\bar{u}(x)\gamma^{\mu}u(x) - \bar{d}(x)\gamma^{\mu}d(x)\right], \tag{3.16}$$

where the γ^{μ} makes sure that J Lorentz transforms as a vector and ensures even parity while the minus sign in the linear combination of flavours preserves the isotriplet nature of the ρ-meson.

When choosing an interpolating field to examine a four-quark $q\bar{q}q\bar{q}$ state we have many possibilities. In [13] they examine the light $q\bar{q}$ pairs which couple to a colour singlet. Ref. [13] examines scalar bound states as either two bound pseudoscalars or two bound vectors, leading to the current

$$j_1(x) = (\bar{q}\Gamma\Lambda^m q)(\bar{q}\Gamma\Lambda^n q)(x)\,, \tag{3.17}$$

where $\Gamma=\gamma_5$ for the pseudoscalar pairs, $\Gamma=\gamma_\mu$ for the vector pairs, and Λ^m is a flavour matrix.

In [25] the states $\sigma(500)$, $f_0(980)$, the isodoublet κ and the $a_0(980)$ are assigned $q\bar{q}q\bar{q}$ interpolating fields given by

$$\begin{aligned}
j_\sigma &= \epsilon_{abc}\epsilon_{dec}(u_a^T C\gamma_5 d_b)(\bar{u}_d\gamma_5 C\bar{d}_e^T)\\[2ex]
j_{f_0} &= \tfrac{\epsilon_{abc}\epsilon_{dec}}{\sqrt{2}}\left[(u_a^T C\gamma_5 s_b)(\bar{u}_d\gamma_5 C\bar{s}_e^T) + u\leftrightarrow d\right]\\[2ex]
j_{a_0} &= \tfrac{\epsilon_{abc}\epsilon_{dec}}{\sqrt{2}}\left[(u_a^T C\gamma_5 s_b)(\bar{u}_d\gamma_5 C\bar{s}_e^T) - u\leftrightarrow d\right]\\[2ex]
j_\kappa &= \epsilon_{abc}\epsilon_{dec}(u_a^T C\gamma_5 d_b)(\bar{q}_d\gamma_5 C\bar{s}_e^T), \quad \bar{q}=\bar{u},\bar{d}\,.
\end{aligned} \tag{3.18}$$

In [15] the association of the current

$$J(x) = \frac{1}{\sqrt{2}}(\bar{u}(x)i\gamma_5 u(x) + \bar{d}(x)i\gamma_5 d(x))\bar{s}(x)i\gamma_5 s(x) + \frac{1}{\sqrt{2}}(\bar{u}(x)u(x) + \bar{d}(x)d(x))\bar{s}(x)s(x) \tag{3.19}$$

to a particle with $J^P=0^+$ is said to be useful because it is divergence free up to the order in which they perform their calculation.

Because there is no experimental evidence which identifies any particular particle as a $q\bar{q}q\bar{q}$ states the choice of current used to analyze the possible existence of such states would seem arbitrary. In fact as long as the current maintains the same quantum numbers as the particle under speculation any current will work, however some currents may turn out to be more useful than others. I will attempt to keep the current as general as possible by not assigning a specific flavour structure. As previously mentioned many of the scalar meson states have a dominant decay into two mesons (usually combinations of π, η or $K\bar{K}$ pairs). These decays imply, or at the very least, provide a reasonable justification for studying the scalar mesons as $q\bar{q}q\bar{q}$ states. While the flavours and the colours of the quarks in my current are kept general, it is important that the current is the product of two colour singlets. In order to keep this a current for a scalar particle I have chosen to use the γ_5 pseudoscalar matrix in the current. The current I will be using in the correlation function is

$$J(x) = (\bar{q}\gamma_5\Lambda q)(\bar{q}\gamma_5\Lambda q)\,. \tag{3.20}$$

3.4 Correlation Function

Suppose we wish to probe hadronic states with the same quantum numbers as the current J. We can define a correlation function $\Pi(q^2)$ as

$$\Pi(q^2) = i\int d^4x\ e^{iq\cdot x}\langle 0|T(J(x)J(0)|0\rangle\,. \tag{3.21}$$

Now if we sum over hadronic states (*i.e.* insert identity)

$$\Pi(q^2) = i\int d^4x\ e^{iq\cdot x}\sum_h \langle 0|J(x)|h\rangle\langle h|J|0\rangle\,, \tag{3.22}$$

we can see that hadronic states which contribute to Π must satisfy $\langle h|J|0\rangle \neq 0$.

In principle we could have vector currents in our correlation function which would add extra degrees of freedom to our problem. We would then have a correlation function that looks like

$$\Pi^{\mu\nu}(q) = \int d^4x e^{iq\cdot x}\langle 0|T(J^\mu(x)J^\nu(0))|0\rangle\,, \tag{3.23}$$

which, because it is a tensor, would have the form

$$\Pi^{\mu\nu}(q) = A(q^2)g^{\mu\nu} + B(q^2)q^\mu q^\nu\,. \tag{3.24}$$

However if J^μ is a conserved current we can apply Noether's theorem to obtain

$$\Pi^{\mu\nu}(q) \equiv \Pi(q^2)\left(q^\mu q^\nu - q^2 g^{\mu\nu}\right)\,, \tag{3.25}$$

which has been reduced to a calculation of $\Pi(q^2)$ which has one degree of freedom. The correlation function, $\Pi(q^2)$, for a scalar current J representing a $q\bar{q}q\bar{q}$ state is the main quantity to be calculated in my thesis. Once this calculation of $\Pi(q^2)$ has been completed, dispersion relations can be applied to $\Pi(q^2)$ and a mass for this theoretical particle can be obtained and compared to those of experimentally observed particles.

3.5 Calculation of Lowest Order Piece of Correlation Function for ρ^0 Meson

The following is a sample calculation, in the chiral limit of massless quarks, of the lowest order piece of the correlation function $\Pi(q^2)$ of the ρ^0 meson. It should be noted that in this calculation we are computing the vacuum expectation value of the perturbative vacuum and not the vacuum for QCD. This is important because extra terms would appear as in the calculations of vacuum expectation values $\langle\Omega|\bar{\psi}(x)\psi(0)|\Omega\rangle$ where $|\Omega\rangle$ is the QCD vacuum state.

We begin by first choosing a current $J^\mu(x)$ for the ρ^0 meson. We will use the current in (3.16):

$$J^\mu \equiv \frac{1}{2}[\bar{u}(x)\gamma^\mu u(x) - \bar{d}(x)\gamma^\mu d(x)] . \tag{3.26}$$

We then have a correlation function $\Pi^{\mu\nu}$ given by

$$\Pi^{\mu\nu}(q) = i\int d^4x e^{iq\cdot x}\langle 0|T(J^\mu(x)J^\nu(0))|0\rangle . \tag{3.27}$$

Because J^μ is a conserved current (*i.e.* $q^\mu\Pi^{\mu\nu} = q^\nu\Pi^{\mu\nu} = 0$) we have

$$\Pi^{\mu\nu}(q) \equiv (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi(q^2) \tag{3.28}$$

$$\Pi(q^2) = -\frac{i}{(D-1)q^2}\int d^D x e^{iq\cdot x}\langle 0|T(J^\mu(x)J^\nu(0))|0\rangle , \tag{3.29}$$

where D is the number of space-time dimensions. Substituting the value of J^μ into the above equation we obtain

$$\Pi(q^2) = -\frac{i}{4q^2(D-1)}\int d^D e^{iq\cdot x}\langle 0|T([\bar{u}(x)\gamma^\mu u(x) - \bar{d}(x)\gamma^\mu d(x)][\bar{u}(0)\gamma_\mu u(0) - \bar{d}(0)\gamma_\mu d(0)])|0\rangle . \tag{3.30}$$

Applying Wick's theorem we get

$$\Pi(q^2) = \frac{i}{4q^2(D-1)}(\gamma^\mu)_{ij}(\gamma_\mu)_{kl}\int d^D x e^{iq\cdot x}\left(\overbrace{u_{j\alpha}(x)\bar{u}_{i\beta}(0)}\cdot\overbrace{u_{l\beta}(0)\bar{u}_{i\alpha}(x)} + (u\rightarrow d)\right) . \tag{3.31}$$

Recall that

$$\overbrace{\psi^A_{i\alpha}(x)\bar{\psi}^B_{j\beta}(y)} = i\delta_{\alpha\beta}\delta_{AB}S^A_{ij}(x-y) = \delta_{\alpha\beta}\delta_{AB}i\int\frac{d^4p}{(2\pi)^4}S^A_{ij}(p)e^{-ip\cdot(x-y)} , \tag{3.32}$$

where $S^A(x-y)$ is the Feynman propagator for QCD and

$$S^A(p) \equiv \frac{1}{\not{p} - m_A} . \tag{3.33}$$

Substitution of (3.32) into (3.31) gives us

$$\Pi(q^2) = -\frac{Ni}{4q^2(D-1)}\int\frac{d^Dp}{(2\pi)^D}\left(Tr[\gamma^\mu S^u(p)\gamma_\mu S^u(p-q)] + (u\rightarrow d)\right) , \tag{3.34}$$

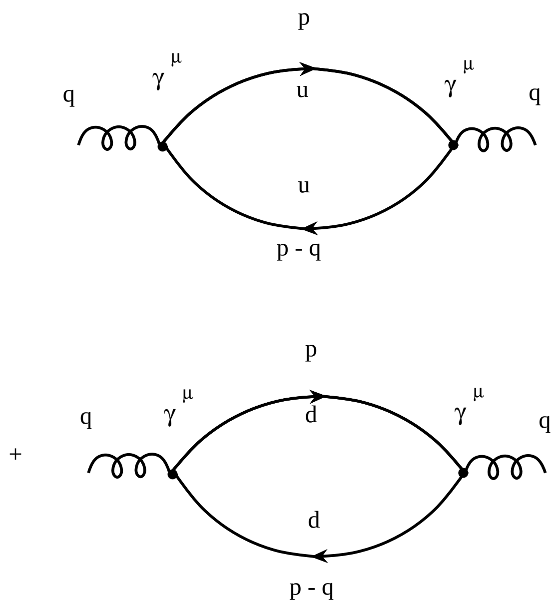

Figure 3.4: Feynman diagram for lowest order pieces of the ρ^0 meson

which corresponds to the Feynman diagrams given in Figure 3.4.

Because we are dealing with up and down quarks we can neglect mass terms of order $\frac{m_{u,d}^2}{q^2}$ and after a little work with trace technology we obtain

$$\Pi(q^2) = \frac{i2N(D-2)}{q^2(D-1)} \int \frac{d^D p}{(2\pi)^D} \frac{p^2 - p \cdot q}{(p^2 + i\eta)((p-q)^2 + i\eta)} \,. \tag{3.35}$$

Using the standard dimensional regularization integral formulas (3.50), (3.51) we get

$$\begin{aligned} \Pi(q^2) =& \frac{N}{8\pi^2} \left(-\frac{q^2}{4\pi\nu^2} \right)^{\epsilon} \frac{\Gamma(1+\epsilon)\Gamma(2+\epsilon)\Gamma(-\epsilon)}{(3+2\epsilon)\Gamma(2+2\epsilon)} \\ =& \frac{1}{8\pi^2} \left(-\frac{1}{\epsilon} + \ln(4\pi) - \gamma - \ln\left(-\frac{q^2}{\nu^2} \right) + \frac{5}{3} \right) , \end{aligned} \tag{3.36}$$

where $D = 4 + 2\epsilon$ and $\gamma \approx .5772$ is the Euler-Mascheroni constant. Note that the divergent terms in (3.36) lead to the combination $-\frac{1}{\epsilon} + \ln(4\pi) - \gamma$, which corresponds to the term that is usually removed in the $\overline{MS}$ renormalization scheme [26]. This calculation results in a momentum-independent divergent term which corresponds to a need for a subtracted dispersion relation. The actual physics lies within the logarithmic term.

3.6 Renormalization and Dimensional Regularization

Once we have the Feynman rules for a quantum field theory we can calculate the Green's functions and S-Matrix elements that we use to determine observable quantities. However when QFT effects are taken into account we find that infinities occur when we calculate diagrams with loops. This is because the loop integration contains a momentum variable that must be integrated over an infinite range. Renormalization theory is extremely important to relativistic quantum field theory and is used to isolate and remove these infinities in order to obtain finite values for the physically measurable quantities of the theory.

Renormalization is not unique to relativistic field theories. For example, consider a finite theory where we have an electron (mass m) moving through a solid. Because the electron will interact with the lattice, any applied external force will act on the electron inside the lattice as if it was a free electron with renormalized mass m^*. In this case the effects of the external force can be measured on both the bare quantity m and the renormalized quantity m^* and compared to each other to determine the renormalization constant. It is easy to see that the renormalization in this case is finite. However in a relativistic field theory the problem of renormalization is not so simple. Because the interaction in the relativistic case corresponds to a divergent loop diagram the renormalization is infinite. There is no way to turn this interaction off and so as a result the bare quantities in a relativistic field theory are infinite as well.

In order to remove these infinities from our measurable quantities in a relativistic theory we must assume that our bare quantities are divergent in such a way that when the interaction is taken into consideration, the infinite renormalization due to the interaction cancels the divergences in the bare quantities to give us finite renormalized quantities. In practice the renormalization programme is quite complicated.

It is important in QCD that we find a method to remove divergences that arise when calculating correlation functions. In QCD and the standard model we use the method of dimensional regularization [27] and renormalization which "preserves gauge invariance to all orders of perturbation theory" [28]. It involves the computation of QCD processes as an analytic function of the dimensionality of space-time D. For appropriate D the loop momentum integrals that we are computing will converge and the expressions can be analytically continued to $D = 4 + 2\epsilon$. After renormalization, the $\epsilon \to 0$ limit for any observable quantity arising from this calculation will have a well defined limit.

As an example let us look at a simple case of the renormalization for a massless quark. Since most of the calculation is done later on in Section 4.3 it will not be necessary to go through every detail of the calculation but instead examine the result contained in (4.44). To obtain the desired result we must expand the Γ functions contained in K given in (4.45). Expanding the terms in K we obtain

$$K = \delta_{\beta\alpha}(g\nu^{\epsilon})^2 C_2(R)\left[3\frac{1}{\epsilon} + 3\gamma - 3 + \mathcal{O}(\epsilon)\right] . \tag{3.37}$$

We still have a divergent term $\frac{1}{\epsilon}$ remaining in our result. Therefore, by relating the renormalized quark field ψ_R to the bare field ψ_0 via

$$\psi_R = Z^{\frac{1}{2}}\psi_0 \,, \tag{3.38}$$

the Green function for the renormalized field will be finite if

$$Z = 1 - (g\nu^{\epsilon})^2 \frac{3C_2(R)}{\epsilon} \,. \tag{3.39}$$

It is important to note that renormalization does not necessarily remove all the divergences in the calculation of correlation functions. Divergences, which are polynomials in momentum q^2, are not a serious problem as we can still determine the real physics of the situation through the use of proper dispersion relations.

3.7 QCD Sum Rules

The most important feature for any physical theory is how well it agrees with experiment. We can probe the hadronic states using correlation functions $\Pi(q^2)$ built out of composite operators, but we need a way of matching these values to experimentally measured quantities. QCD sum rules allows us to make this connection via dispersion relations of the form:

$$\Pi(q^2) = \int \frac{Im(\Pi(t))}{(t+q^2)} dt \,, \tag{3.40}$$

where the left hand side represents the theoretical calculation and $Im(\Pi(t))$ on the right hand side represents the experimental quantities. For example in the ρ meson case $Im(\Pi(t)) \approx \sigma(e^+e^- \rightarrow$ hadrons). Such dispersion relations must be differentiated with respect to q^2 or further transformed (e.g. Borel Transforms) in order to be of practical value. This is the essence of QCD sum rules, the method having been developed in [21].

3.8 Renormalization of Composite Operators

In my particular calculation I am specifically calculating a correlation function involving composite operators which are local monomials of fields and their derivatives. (*i.e.* $\bar{\psi}\gamma_\mu\psi, \phi^2, \phi\partial^2\phi$). The principal quantity I wish to calculate,

$$\Pi(q^2) = \int d^4x e^{iq\cdot x}\langle 0|T(J(x)J(0)|0\rangle \,, \tag{3.41}$$

contains the current operator $J(x)$ which, because we are looking at $q\bar{q}q\bar{q}$ states, will be a current containing pairs of the composite operators $q\bar{q}$.

The presence of composite operators adds a complication to the renormalization process necessary for obtaining the observable quantities of the theory. One must take into account the divergences resulting from the mixing of composite operators in addition to the standard renormalization process.

We shall refer to J_R as the current operator J renormalized such that the divergences due to composite operators are removed. We then let

$$J_R = Z_J \mathbf{O}\,, \tag{3.42}$$

where the Z_J's are renormalization matrices and the $\mathbf{O}$'s are the set of operators that mix with J under renormalization (see [29] for a discussion of the renormalization of gauge invariant composite operators). We can also write J_R in terms of elements of the matrix Z_J as

$$J_R = Z_{JJ} J_B + \sum_{\mathbf{O}} Z_{J\mathbf{O}} \mathbf{O}\,, \tag{3.43}$$

where J_B is the bare quantity J before composite operator renormalization. For lowest order in α_s^0, $J_R = J_B$ Which implies

$$Z_{JJ} = Z_{JJ}^{(0)} + Z_{JJ}^{(1)} + \cdots \tag{3.44}$$

with $Z_{JJ}^{(0)} = 1$ and $Z_{JJ}^{(1)} = \mathcal{O}(\alpha)$ and

$$Z_{J\mathbf{O}} = Z_{J\mathbf{O}}^{(1)} + \cdots \tag{3.45}$$

where $Z_{J\mathbf{O}}^{(1)} = \mathcal{O}(\alpha)$. Also we can expand the matrix element of the bare currents as

$$\langle 0|T(J_B(x)J_B(0))|0\rangle = \langle 0|T(J_B(x)J_B(0))|0\rangle^{(0)} + \langle 0|T(J_B(x)J_B(0))|0\rangle^{(1)} + \cdots\,, \tag{3.46}$$

where $\langle 0|T(J_B(x)J_B(0))|0\rangle^{(0)}$ refers to the lowest order (α^0) term and $\langle 0|T(J_B(x)J_B(0))|0\rangle^{(1)}$ refers to the next order (α^1) term.

The quantity we wish to calculate to order α_s is

$$\Pi(q^2) = \langle 0|T(J_R(x)J_R(0))|0\rangle = \langle 0|T(J_R(x)J_R(0))|0\rangle^{(0)} + \langle 0|T(J_R(x)J_R(0))|0\rangle^{(1)}\,, \tag{3.47}$$

where $\langle 0|T(J_R(x)J_R(0))|0\rangle^{(0)}$ is the lowest order piece ($(\alpha_s)^0$) and $\langle 0|T(J_R(x)J_R(0))|0\rangle^{(1)}$ is the piece proportional to α_s. Using the above equations we get for lowest order in α_s:

$$\langle 0|T(J_R(x)J_R(0))|0\rangle^{(0)} = \langle 0|T(J_B(x)J_B(x))|0\rangle^{(0)}\,, \tag{3.48}$$

which corresponds to the Feynman diagram in Figure 3.5.

The α_s piece is

$$\begin{aligned}\langle 0|T(J_R(x)J_R(0))|0\rangle^{(1)} =& \langle 0|T(J_B(x)J_B(0))|0\rangle^{(1)} + 2Z_{JJ}^{(1)}\langle 0|T(J_B(x)J_B(0))|0\rangle^{(0)} \\ &+ \sum_{\mathbf{O}} Z_{J\mathbf{O}}^{(1)}\left[\langle 0|T(J_B(x)\mathbf{O}(0))|0\rangle^{(0)} + \langle 0|T(\mathbf{O}(x)J_B(0))|0\rangle^{(0)}\right]\,,\end{aligned} \tag{3.49}$$

which corresponds to the Feynman diagrams displayed in Figure 3.6. The first two diagrams in Figure 3.6 correspond to the first term in (3.49) and the remaining diagrams in Figure 3.6 correspond

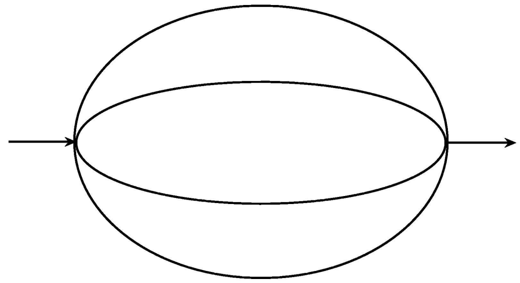

Figure 3.5: Feynman diagram for lowest order α^0 process

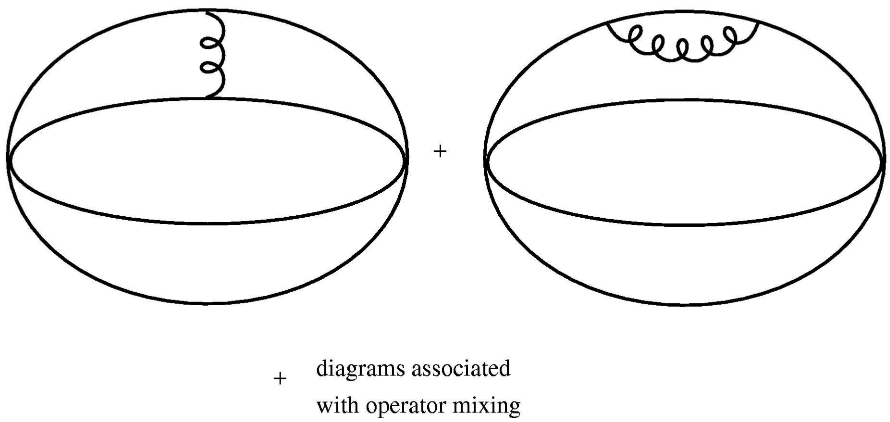

Figure 3.6: Feynman diagram for order α

to the terms of order α_s that are associated with operator mixing. As a result, the operator-mixing diagrams are one-loop lower than the first two diagrams.

I will be addressing the the first two Feynman diagrams above. The remaining terms of order α_s remove the divergence due to the composite operators, however they contain operators resulting from the renormalization process for which we require extra information to calculate their contribution. Renormalization of four-quark operators has only been performed for a single flavour, and results in a set of ten mixed operators [30]. The extension of [30] to three flavours is beyond the scope of this thesis. While my calculation does not encompass all pieces in the first order α_s contribution, it will nonetheless be a significant addition to the lowest order version of this calculation and will still be a significant challenge in itself.

3.9 Feynman Integrals

In order to perform the calculation of the lowest and next order terms in the correlation function, we will need to utilize the following general formulae for evaluating the one-loop Feynman integrals taken from [31]:

$$\begin{aligned}
I(p;r,s) =& \frac{1}{\nu^{2\epsilon}} \int \frac{d^D q}{(2\pi)^D} \frac{1}{[q^2+i\eta]^r[(q-p)^2+i\eta]^s} \\
=& \frac{i}{(4\pi)^2} \left(-\frac{p^2}{4\pi\nu^2}\right)^\epsilon \frac{1}{p^{2(r+s-2)}} \frac{\Gamma(2-r+\epsilon)\Gamma(2-s+\epsilon)\Gamma(r+s-2-\epsilon)}{\Gamma(r)\Gamma(s)\Gamma(4-r-s+2\epsilon)} \\
=& \frac{i}{(4\pi)^2} \left(-\frac{p^2}{4\pi\nu^2}\right)^\epsilon \frac{1}{p^{2(r+s-2)}} G_1(r,s)
\end{aligned} \tag{3.50}$$

$$\begin{aligned}
I^\mu(p;r,s) =& \frac{1}{\nu^{2\epsilon}} \int \frac{d^D q}{(2\pi)^D} \frac{q^\mu}{[q^2+i\eta]^r[(q-p)^2+i\eta]^s} \\
=& \frac{i}{(4\pi)^2} \left(-\frac{p^2}{4\pi\nu^2}\right)^\epsilon \frac{1}{p^{2(r+s-2)}} p^\mu \frac{\Gamma(3-r+\epsilon)\Gamma(2-s+\epsilon)\Gamma(r+s-2-\epsilon)}{\Gamma(r)\Gamma(s)\Gamma(5-r-s+2\epsilon)} \\
=& \frac{i}{(4\pi)^2} \left(-\frac{p^2}{4\pi\nu^2}\right)^\epsilon \frac{1}{p^{2(r+s-2)}} p^\mu G_2(r,s)
\end{aligned} \tag{3.51}$$

$$\begin{aligned}
I^{\mu\nu}(p;r,s) =& \frac{1}{\nu^{2\epsilon}} \int \frac{d^D q}{(2\pi)^D} \frac{q^\mu q^\nu}{[q^2+i\eta]^r[(q-p)^2+i\eta]^s} \\
=& \frac{i}{(4\pi)^2} \left(-\frac{p^2}{4\pi\nu^2}\right)^\epsilon \frac{1}{p^{2(r+s-2)}} \left[g^{\mu\nu} p^2 \frac{\Gamma(3-r+\epsilon)\Gamma(3-s+\epsilon)\Gamma(r+s-3-\epsilon)}{2\Gamma(r)\Gamma(s)\Gamma(6-r-s+2\epsilon)} \right. \\
& \left. + p^\mu p^\nu \frac{\Gamma(4-r+\epsilon)\Gamma(2-s+\epsilon)\Gamma(r+s-2-\epsilon)}{\Gamma(r)\Gamma(s)\Gamma(6-r-s+2\epsilon)} \right]
\end{aligned} \tag{3.52}$$

where the notation $G_1(r,s)$ and $G_2(r,s)$ are used to represent the Γ function piece of the integrals. The integrals do not carry a mass term as a result of working in the chiral limit. From here on the limit $\eta \to 0^+$ is implicit. While all these formulae are standard results it will be helpful to derive the basic formula for

$$I = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(p-k)^2} \,. \tag{3.53}$$

We begin by using the Feynman parameterization [31]

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax+b(1-x)]^2} \,, \tag{3.54}$$

with $a=(p-k)^2$ and $b=k^2$ to get

$$I = \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{[x(p-k)^2+(1-x)k^2]^2} \,. \tag{3.55}$$

Next we look at the denominator piece

$$\begin{aligned}
x(p-k)^2+(1-x)k^2 &= x(p^2+k^2-2p\cdot k)+(1-x)k^2 \\
&= k^2+xp^2-2xp\cdot k \,,
\end{aligned} \tag{3.56}$$

and let $q = k - xp$. After some rearranging we obtain

$$I = \int_0^1 dx \int \frac{d^D q}{(2\pi)^D} \frac{1}{[q^2 + p^2 x(1-x)]^2} \,. \tag{3.57}$$

It will be helpful to derive the basic dimensional regularization formula as $\eta \to 0$

$$R(D, \alpha, \beta) \equiv \int \frac{d^D k}{(2\pi)^D} \frac{[k^2]^\alpha}{[k^2 - a^2 + i\eta]^\beta} \,. \tag{3.58}$$

If we look at $k^2 = k_0^2 - (\vec{k})^2$ when solving this integral we can see that there are poles at $k_0 = \pm\sqrt{(\vec{k})^2 + a^2 - i\eta}$. This problem can be solved by performing a Wick rotation and integrating along the imaginary k_0 axis as seen in Figure 3.7. The transformation $k_0 \to i\overline{k_0}$ where $\overline{k_0}$ is real

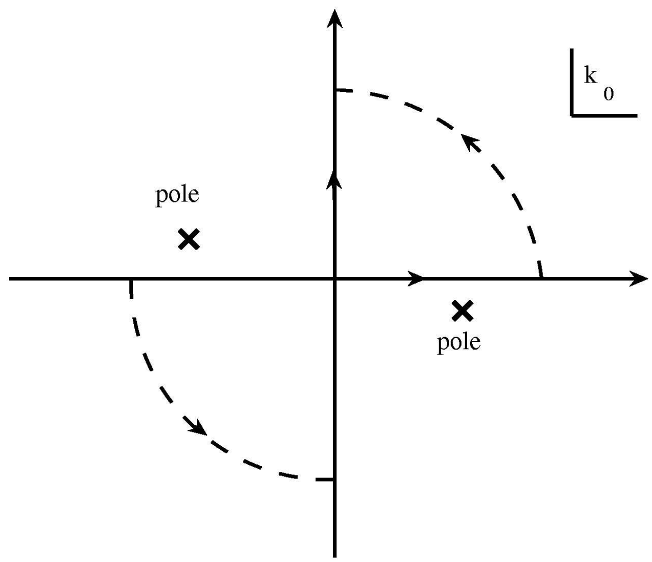

Figure 3.7: Wick Rotation

gives us

$$k^2 = k_0^2 - \vec{k}^2 = -[(\overline{k_0})^2 + (\vec{k})^2] = -k_E^2 \,, \tag{3.59}$$

where k_E is a typical Euclidean vector whose square is real. Noting that $dk_0 \to i\overline{dk_0}$ we obtain an R of the form

$$R(D, \alpha, \beta) = i \int \frac{d\overline{k_0} \quad d^{D-1}k}{(2\pi)^D} \frac{[-k_E^2]^\alpha}{[-(k_E^2 + a^2)]^\beta} = i(-1)^{\alpha-\beta} \int \frac{d^D k_E}{(2\pi)^D} \frac{(k_E^2)^\alpha}{(k_E^2 + a^2)^\beta} \,, \tag{3.60}$$

where $k_E^2 > 0$ and we have let $\eta \to 0$. If we denote the solid angle in this Euclidean space as $d\Omega_k$

$$d^D k \equiv k^{D-1} dk d\Omega_k \tag{3.61}$$

(3.60) becomes

$$R(D, \alpha, \beta) = i(-1)^{\alpha-\beta} \int_0^\infty \frac{dk_E}{(2\pi)^D} \frac{k_E^{D-1}(k_E^2)^\alpha}{(k_E^2 + a^2)^\beta} \int d\Omega_k \,, \tag{3.62}$$

as seen in Eq. (C.31) of [31], where the integral over $d\Omega_k$ is the area of a sphere in this space with $D = 4+2\epsilon$ given by [31]

$$\int d\Omega_k = \frac{2\pi^{2+\epsilon}}{\Gamma(2+\epsilon)} \,. \tag{3.63}$$

Hence (3.62) becomes

$$R(D,\alpha,\beta) = i(-1)^{\alpha-\beta}\frac{2\pi^{2+\epsilon}}{\Gamma(2+\epsilon)}\int_0^\infty \frac{dk_E}{(2\pi)^{4+2\epsilon}}\frac{(k_E)^{2\alpha}(k_E)^{3+2\epsilon}}{(k_E^2+a^2)^\beta} \,. \tag{3.64}$$

Now we make the substitution $k_E = ar \Rightarrow dk_E = adr$ to obtain

$$R(D,\alpha,\beta) = i(-a^2)^{\alpha-\beta}a^{4+2\epsilon}\frac{2\pi^{2+\epsilon}}{\Gamma(2+\epsilon)}\frac{1}{(2\pi)^{4+2\epsilon}}\int_0^\infty dr\frac{r^{2\alpha}r^{3+2\alpha}}{(r^2+1)^\beta} \,. \tag{3.65}$$

Using a standard integral formula Eq. (3.251.2) of [32]

$$\int_0^\infty dx \quad x^{\mu-1}(1+x^2)^{\nu-1} = \frac{1}{2}B\left[\frac{\mu}{2}, 1-\nu-\frac{\mu}{2}\right] , \tag{3.66}$$

where $B[z,w]$ is the beta function whose value in terms of Γ functions is given in the Appendix (A.29), we obtain the following value for $R(D,\alpha,\beta)$:

$$R(D,\alpha,\beta) = i(-a^2)^{\alpha-\beta+2}\left(\frac{a^2}{4\pi}\right)^\epsilon \frac{\Gamma(2+\alpha+\epsilon)\Gamma(\beta-\alpha-2-\epsilon)}{\Gamma(\beta)\Gamma(2+\epsilon)} \,. \tag{3.67}$$

We can now finish the derivation of (3.53) by substituting the value of $R(D,\alpha,\beta)$ evaluated for $D = 4+2\epsilon$,$\alpha = 0$, $\beta = 2$ and $a^2 = -p^2x(1-x)$ into (3.57) to obtain

$$I = \frac{i}{(4\pi)^2}\int_0^\infty dx\left[\frac{-p^2x(1-x)}{4\pi}\right]^\epsilon \frac{\Gamma(-\epsilon)}{\Gamma(2)} \,. \tag{3.68}$$

Using the definition of the Beta function given in the Appendix (A.29) and the recursion relations $\Gamma(n) = (n-1)!\Gamma(n-1)$ and $\Gamma(1) = 1$ we obtain

$$\begin{aligned} I &= \frac{i}{(4\pi)^2}\left(-\frac{p^2}{4\pi}\right)^\epsilon \Gamma(-\epsilon)B[1+\epsilon, 1+\epsilon] \\ &= \frac{i}{(4\pi)^2}\left(-\frac{p^2}{4\pi}\right)^\epsilon \frac{(\Gamma(1+\epsilon))^2\Gamma(-\epsilon)}{\Gamma(2+2\epsilon)} , \end{aligned} \tag{3.69}$$

which is our final result for I. Now that we have these formulae for the basic one-loop Feynman integrals we only have one more type of integral to examine before we can proceed to calculate the lowest and next order pieces of the correlation function $\Pi(q^2)$.

3.10 Three-Loop Feynman Integrals

In the process of calculating the correlation function $\Pi(q^2)$ for a $q\bar{q}q\bar{q}$ pair we will run into several three-loop Feynman integrals for which there are no general formulae. While there is no one particular strategy for handling all three-loop Feynman integrals, it is important to examine several of the three-loop integrals that will appear in the calculation.

The first type of three-loop Feynman integral is the integral that can be separated into three distinct one-loop integrals, using substitution, and evaluated one at a time. For example an integral of the form

$$I = \frac{1}{(\nu)^{6\epsilon}} \int\int\int \frac{d^Dq_1 d^Dq_2 d^Dq_3}{(2\pi)^{3D}} \frac{1}{(q_1^2+i\eta)(q_2^2+i\eta)(q_3^2+i\eta)[(q_3+q_1+q_2-p)^2+i\eta]} \tag{3.70}$$

can be manipulated by using the substitution $-l = q_1+q_2-p$ and bringing any terms not depending on q_3 outside the q_3 integral,

$$I = \frac{1}{\nu^{2\epsilon}} \int \frac{d^Dq_1}{(2\pi)^D} \frac{1}{(q_1^2+i\eta)} \frac{1}{\nu^{2\epsilon}} \int \frac{d^Dq_2}{(2\pi)^D} \frac{1}{(q_2^2+i\eta)} \frac{1}{\nu^{2\epsilon}} \int \frac{d^Dq_3}{(2\pi)^D} \frac{1}{(q_3^2+i\eta)[(q_3-l)^2+i\eta]}\,, \tag{3.71}$$

where the d^Dq_3 integral is of the form of a standard one-loop integral. Evaluating the d^Dq_3 integral we obtain

$$I = \frac{iG(1,1)}{(4\pi)^2}\left(\frac{-1}{4\pi\nu^2}\right)^{\epsilon} \frac{1}{\nu^{2\epsilon}} \int \frac{d^Dq_1}{(2\pi)^D} \frac{1}{(q_1^2+i\eta)} \frac{1}{\nu^{2\epsilon}} \int \frac{d^Dq_2}{(2\pi)^D} \frac{1}{(q_2^2+i\eta)[(q_2+q_1-p)^2]^{-\epsilon}} \tag{3.72}$$

where

$$G(r,s) \equiv \frac{\Gamma(2-r+\epsilon)\Gamma(2-s+\epsilon)\Gamma(r+s-2-\epsilon)}{\Gamma(r)\Gamma(s)\Gamma(4-r-s+2\epsilon)}\,. \tag{3.73}$$

Again using a substitution $-\xi = q_1 - p$ we can now manipulate the q_2 integral into one of our standard Feynman integrals

$$\begin{aligned} I &= \frac{i}{(4\pi)^2}\left(\frac{-1}{4\pi\nu^2}\right)^{\epsilon} \frac{1}{\nu^{2\epsilon}} \int \frac{d^Dq_1}{(2\pi)^D} \frac{1}{(q_1^2+i\eta)} \frac{1}{\nu^{2\epsilon}} \int \frac{d^Dq_2}{(2\pi)^D} \frac{1}{(q_2^2+i\eta)[(q_2-\xi)^2]^{-\epsilon}} \\ &= -\frac{G(1,1)G(1,-\epsilon)}{(4\pi)^4}\left(\frac{-1}{4\pi\nu^2}\right)^{2\epsilon} \frac{1}{\nu^{2\epsilon}} \int \frac{d^Dq_1}{(2\pi)^D} \frac{1}{(q_1^2+i\eta)[\xi^2]^{-2\epsilon-1}}\,. \end{aligned} \tag{3.74}$$

The final q_1 integral is already in the form of a standard Feynman one-loop integral and so we obtain for our final result

$$\begin{aligned} I &= -\frac{G(1,1)G(1,-\epsilon)}{(4\pi)^4}\left(\frac{-1}{4\pi\nu^2}\right)^{2\epsilon} \frac{1}{\nu^{2\epsilon}} \int \frac{d^Dq_1}{(2\pi)^D} \frac{1}{(q_1^2+i\eta)[(q_1-p)^2]^{-2\epsilon-1}} \\ I &= -\frac{iG(1,1)G(1,-\epsilon)G(1,-2\epsilon-1)}{(4\pi)^6}\left(\frac{1}{4\pi\nu^2}\right)^{3\epsilon} (-p^2)^{3\epsilon+2}\,. \end{aligned} \tag{3.75}$$

The majority of three-loop integrals in this thesis calculation are of this form. However there is one more type of three loop integral in which the different loop momenta mix in such a way as to make it impossible to separate them into individual one-loop integrals. This three-loop integral is of the form

$$I' = \frac{1}{\nu^{6\epsilon}} \int\int\int \frac{d^Dq\, d^Dk_1 d^Dk_2}{(2\pi)^{3D}} \frac{1}{k_1^2k_2^2(k_1+q)^2(k_2+q)^2(k_1-k_2)^2q^{-4}(q-p)^{-2-2\epsilon}}\,. \tag{3.76}$$

Rearranging this integral, we can see, will not lead to a standard Feynman integral because all three loop momenta are tied to each other in the denominator. We can however separate this integral somewhat. By isolating the pieces of (3.76) which contain q and p but not k_1 or k_2 we can break the problem of solving this three-loop integral down to one in which we have to solve a two-loop integral and a standard one-loop Feynman integral

$$I' = \frac{1}{\nu^{2\epsilon}} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2)^{-2}[(q-p)^2]^{-1-\epsilon}} \frac{1}{(\nu)^{4\epsilon}} \int\int \frac{d^D k_1 d^D k_2}{(2\pi)^{2D}} \frac{1}{k_1^2 k_2^2 (k_1+q)^2 (k_2+q)^2 (k_1-k_2)^2} \tag{3.77}$$

With special thanks to Ian Blockland for his guidance at this point, we will first find a recursion relationship for the general two loop integral using the integration by parts technique outlined in [33]:

$$I_2'(a_1,a_2,a_3,a_4,a_5) = \frac{1}{(\nu)^{4\epsilon}} \int\int \frac{d^D k_1 d^D k_2}{(2\pi)^{2D}} \frac{1}{(k_1^2)^{a_1}(k_2^2)^{a_2}(k_1+q)^{2a_3}(k_2+q)^{2a_4}(k_1-k_2)^{2a_5}} . \tag{3.78}$$

We can assume that dimensional regularization adjusts the dimensions of our problem in such a way that the integral is well defined. This being the case we can apply the divergence theorem to (3.78). Since the integral is well defined, the surface integration at infinity goes to zero and we obtain the identity

$$0 = \int d^D k_1 d^D k_2 \frac{\partial}{\partial k_1^\mu}\left[(k_1-k_2)^\mu \frac{1}{k_1^{2a_1} k_2^{2a_2}(k_1+q)^{2a_3}(k_2+q)^{2a_4}(k_1-k_2)^{2a_5}}\right] \tag{3.79}$$

After taking the derivative we obtain

$$\begin{aligned} 0 = &\int d^D k_1 d^D k_2 \frac{D}{k_1^{2a_1} k_2^{2a_2}(k_1+q)^{2a_3}(k_2+q)^{2a_4}(k_1-k_2)^{2a_5}} \\ &- 2\int d^D k_1 d^D k_2 \frac{(k_1-k_2)^\mu}{k_1^{2a_1} k_2^{2a_2}(k_1+q)^{2a_3}(k_2+q)^{2a_4}(k_1-k_2)^{2a_5}}\left[\frac{a_1 k_1^\mu}{k_1^2} + \frac{a_3(k_1+q)^\mu}{(k_1+q)^2} + \frac{a_5(k_1-k_2)^\mu}{(k_1-k_2)^2}\right] \end{aligned} \tag{3.80}$$

$$\begin{aligned} 0 =& D I_2'(a_1,a_2,a_3,a_4,a_5) - 2\int d^D k_1 d^D k_2 \frac{1}{k_1^{2a_1} k_2^{2a_2}(k_1+q)^{2a_3}(k_2+q)^{2a_4}(k_1-k_2)^{2a_5}} \\ &\times\left[a_1 - a_1\frac{k_2\cdot k_1}{k_1^2} + a_3\frac{k_1^2 + k_1\cdot q - k_2\cdot k_1 - k_2\cdot q}{(k_1+q)^2} + a_5\right] . \end{aligned} \tag{3.81}$$

The following substitution

$$d_1 \equiv k_1^2, \quad d_2 \equiv k_2^2, \quad d_3 \equiv (k_1+q)^2, \quad d_4 \equiv (k_2+q)^2, \quad d_5 \equiv (k_1-k_2)^2 \tag{3.82}$$

is used to obtain

$$\begin{aligned} 0 =& D I_2'(a_1,a_2,a_3,a_4,a_5) - \int \frac{d^D k_1 d^D k_2}{(2\pi)^{2D}} \frac{1}{d_1^{a_1} d_2^{a_2} d_3^{a_3} d_4^{a_4} d_5^{a_5}} \\ &\times\left[(a_1+a_3+2a_5) - a_1\left(\frac{d_2}{d_1} - \frac{d_5}{d_1}\right) + a_3\left(\frac{d_5}{d_3} - \frac{d_4}{d_3}\right)\right] . \end{aligned} \tag{3.83}$$

Now replacing the d_n's with their values in terms of k_1 and k_2 we obtain the following recursion relationship:

$$
\begin{aligned}
I_2'(a_1,a_2,a_3,a_4,a_5) =& \frac{1}{(D-a_1-a_3-2a_5)}\Bigg(a_1\Bigg[I_2'(a_1+1,a_2,a_3,a_4,a_5-1)-I_2'(a_1+1,a_2-1,a_3,a_4,a_5)\Bigg] \\
&+a_3\Bigg[I_2'(a_1,a_2,a_3+1,a_4,a_5-1)-I_2'(a_1,a_2,a_3+1,a_4-1,a_5)\Bigg]\Bigg).
\end{aligned}
\tag{3.84}
$$

Now that we have found a recursion relationship for (3.78), we can apply this to solving (3.76) by observing that the two loop Feynman integral in (3.77) is simply $I_2'(1,1,1,1,1)$. We must calculate this value by applying (3.84). Remember that our dimension $D = 4+2\epsilon$, so that

$$
I_2'(1,1,1,1,1) = \frac{1}{2\epsilon}\Bigg(I_2'(2,1,1,1,0)-I_2'(2,0,1,1,1)+I_2'(1,1,2,1,0)-I_2'(1,1,2,0,1)\Bigg). \tag{3.85}
$$

All the integrals on the right hand side of (3.85) are simple two-loop Feynman integrals that can be solved by either, separating them into two one-loop integrals, or through a simple substitution and iteration. The values of the integrals on the right hand side of (3.85) can be found in the appendix. Using these values we obtain a result for $I_2'(1,1,1,1,1)$

$$
\begin{aligned}
I_2'(1,1,1,1,1) =& \frac{-1}{\epsilon(4\pi)^4}\left(-\frac{1}{4\pi\nu^2}\right)^{2\epsilon} q^{4\epsilon-2}\Bigg[\frac{\Gamma(1+\epsilon)^3\Gamma(-\epsilon)\Gamma(\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+2\epsilon)\Gamma(1+2\epsilon)} \\
&-\frac{\Gamma(1+\epsilon)^2\Gamma(-\epsilon)\Gamma(\epsilon)\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)\Gamma(1+3\epsilon)\Gamma(2+2\epsilon)}\Bigg].
\end{aligned}
\tag{3.86}
$$

Now substituting this value into (3.77) we obtain

$$
\begin{aligned}
I' =& \frac{-1}{\epsilon(4\pi)^4}\left(-\frac{1}{4\pi\nu^2}\right)^{2\epsilon}\Bigg[G(1,1)G(2,1)-G(1,1)G(2,1-\epsilon)\Bigg] \\
&\times\frac{1}{\nu^{2\epsilon}}\int\frac{d^Dq}{(2\pi)^D}\frac{q^{4\epsilon-2}}{(q^2)^{-2}[(q-p)^2]^{-1-\epsilon}}
\end{aligned}
\tag{3.87}
$$

Finally we solve the last one-loop Feynman Integral in q to obtain the result for our three-loop Feynman integral I'

$$
I' = \frac{-i}{\epsilon(4\pi)^6}\left(-\frac{1}{4\pi\nu^2}\right)^{3\epsilon}G(-2\epsilon-1,-1-\epsilon)\Bigg[G(1,1)G(2,1)-G(1,1)G(2,1-\epsilon)\Bigg](p^2)^{4\epsilon+4}, \tag{3.88}
$$

where $G(r,s)$ is defined in (3.73).

Chapter 4

Calculation of Correlation Function $\Pi(q^2)$ for a $q\bar{q}q\bar{q}$ State

The primary focus of this thesis is the analytic perturbative calculation, in the chiral limit, of the correlation function for a $q\bar{q}q\bar{q}$ state to first order in α_s. In the process of calculating $\Pi(q^2)$ we will take into account only those terms that correspond to connected Green's functions. We begin with the quantity

$$\Pi(q^2) = i\int d^4x e^{iq\cdot x}\langle 0|T(J(x)\exp\left[i\int d^4w\mathcal{L}_{int}(w)\right]J(0))|0\rangle\,, \tag{4.1}$$

where

$$\mathcal{L}_{int} \equiv \frac{g}{2}\bar{q}(w)\lambda^a\gamma^\mu q(w)A_a^\mu(w) \tag{4.2}$$

is the piece of the QCD Lagrangian that corresponds to the quark-gluon interaction. Expanding the interaction exponential in (4.1) we obtain

$$\begin{aligned}\Pi(q^2) =& i\int d^4x e^{iq\cdot x}\langle 0|T(J(x)\left[1+\frac{ig}{2}\lambda^a\int d^4w\bar{q}(w)\gamma^\mu A_\mu^a q(w)\right.\\ &\left.-\frac{g^2}{8}\lambda^a\lambda^b\int d^4w\int d^4z\bar{q}(w)\gamma^\mu A_\mu^a(w)q(w)\bar{q}(z)\gamma^\nu A_\nu^b(z)q(z)\right]J(0))|0\rangle\\ =& i\int d^4x e^{iq\cdot x}\langle 0|T(J(x)J(0))|0\rangle - \frac{g^2}{8}\lambda^a\lambda^b\int d^4x e^{iq\cdot x}\\ &\times\int d^4w\int d^4z\langle 0|T(J(x)\bar{q}(w)\gamma^\mu A_\mu^a(w)q(w)\bar{q}(z)\gamma^\nu A_\nu^b(z)q(z)J(0))|0\rangle\,.\end{aligned} \tag{4.3}$$

The second piece in $\Pi(q^2)$ that contains only one gluon field $A_\mu(w)$ vanishes because without another gluon field in the expression, every permutation resulting from the application of Wick's theorem results in an uncontracted gluon field causing this term to equal zero. Our expression for $\Pi(q^2)$ becomes

$$\begin{aligned}\Pi(q^2) =& i\int d^4x e^{iq\cdot x}\langle 0|T(J(x)J(0))|0\rangle - i\frac{g^2}{8}\lambda^a\lambda^b\int d^4x e^{iq\cdot x}\\ &\times\int d^4w\int d^4z\langle 0|T(J(x)\bar{q}(w)\gamma^\mu A_\mu^a(w)q(w)\bar{q}(z)\gamma^\nu A_\nu^b(z)q(z)J(0))|0\rangle\\ =&\Pi_1(q^2) - \Pi_2(q^2)\,.\end{aligned} \tag{4.4}$$

The calculation of $\Pi(q^2)$ has been decomposed into the calculation of two quantities, the lowest order piece

$$\Pi_1(q^2) = i\int d^4x e^{iq\cdot x}\langle 0|T(J(x)J(0))|0\rangle\,, \tag{4.5}$$

and the next highest order piece

$$\begin{aligned}\Pi_2(q^2) =& i\frac{g^2}{8}\lambda^a\lambda^b\int d^4x e^{iq\cdot x}\\ &\times\int d^4w\int d^4z\langle 0|T(J(x)\bar{q}(w)\gamma^\mu A^a_\mu(w)q(w)\bar{q}(z)\gamma^\nu A^b_\nu(z)q(z)J(0))|0\rangle\,.\end{aligned} \tag{4.6}$$

The lowest order piece $\Pi_1(q^2)$ is calculated in Section 4.1 while the next highest order piece $\Pi_2(q^2)$ is calculated in Sections 4.2 to 4.6.

4.1 Lowest Order Calculation

Now that a current $J(x)$ has been chosen [see Eq. 3.20] the correlation function $\Pi(q^2)$ can be examined at the lowest order. In order to calculate

$$\Pi_1(q^2) = i\int d^4x e^{iqx}\langle 0|T(J(x)J(0))|0\rangle\,, \tag{4.7}$$

we must first examine the $\langle 0|T(J(x)J(0))|0\rangle$ piece. By substituting the chosen current $J(x) = (\bar{q}\gamma_5\Lambda q)(\bar{q}\gamma_5\Lambda q)$ into the above expression we obtain

$$\begin{aligned}\langle 0|T(J(x)J(0))|0\rangle =& \langle 0|T[(\bar{q}_{i\alpha A}(\gamma_5)_{ij}(\Lambda)_{AB}\; q_{j\alpha B})(\bar{q}_{k\beta C}(\gamma_5)_{kl}(\Lambda)_{CD}\; q_{l\beta D})(x)\\ &\times(\bar{q}_{m\theta E}(\gamma_5)_{mn}(\Lambda)_{EF}\; q_{n\theta F})(\bar{q}_{o\phi G}(\gamma_5)_{op}(\Lambda)_{GH}\; q_{p\phi H})(0)]|0\rangle\end{aligned} \tag{4.8}$$

By applying Wick's theorem we obtain

$$\begin{aligned}\langle 0|T(J(x)J(0))|0\rangle =& (\gamma_5)_{ij}(\gamma_5)_{kl}(\gamma_5)_{mn}(\gamma_5)_{op}\\ &\times\Bigg[f_1 S^A_{ni}(x)S^B_{jm}(x)S^C_{pk}(x)S^D_{lo}(x) + f_2 S^A_{ni}(x)S^B_{jo}(x)S^C_{pk}(x)S^D_{lm}(x)\\ &+\quad f_3 S^A_{pi}(x)S^B_{jm}(x)S^C_{nk}(x)S^D_{lo}(x) + f_4 S^A_{pi}(x)S^B_{jo}(x)S^C_{nk}(x)S^D_{lm}(x)\Bigg]\,,\end{aligned} \tag{4.9}$$

where

$$
\begin{aligned}
f_1 \equiv &(\delta_{\alpha\theta})^2(\delta_{\beta\phi})^2\delta_{AF}\delta_{BE}\delta_{CH}\delta_{DG}(\Lambda)_{AB}(\Lambda)_{CD}(\Lambda)_{EF}(\Lambda)_{GH} \\
= &(N_C)^2(\Lambda)_{FE}(\Lambda)_{HG}(\Lambda)_{EF}(\Lambda)_{GH} \\
= &(N_C)^2(Tr[(\Lambda)^2])^2
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
f_2 \equiv &\delta_{\alpha\theta}\delta_{\alpha\phi}\delta_{\beta\phi}\delta_{\beta\theta}\delta_{AF}\delta_{BG}\delta_{CH}\delta_{DE}(\Lambda)_{AB}(\Lambda)_{CD}(\Lambda)_{EF}(\Lambda)_{GH} \\
= &(\delta_{\theta\phi})^2(\Lambda)_{FG}(\Lambda)_{HE}(\Lambda)_{EF}(\Lambda)_{GH} \\
= &N_C Tr[(\Lambda)^4]
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
f_3 \equiv &\delta_{\alpha\phi}\delta_{\alpha\theta}\delta_{\beta\theta}\delta_{\beta\phi}\delta_{AH}\delta_{BE}\delta_{CF}\delta_{DG}(\Lambda)_{AB}(\Lambda)_{CD}(\Lambda)_{EF}(\Lambda)_{GH} \\
= &(\delta_{\phi\theta})^2(\Lambda)_{HE}(\Lambda)_{FG}(\Lambda)_{EF}(\Lambda)_{GH} \\
= &N_C Tr[(\Lambda)^4]
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
f_4 \equiv &(\delta_{\alpha\phi})^2(\delta_{\beta\theta})^2\delta_{AH}\delta_{BG}\delta_{CF}\delta_{DE}(\Lambda)_{AB}(\Lambda)_{CD}(\Lambda)_{EF}(\Lambda)_{GH} \\
= &(N_C)^2(\Lambda)_{HG}(\Lambda)_{FE}(\Lambda)_{EF}(\Lambda)_{GH} \\
= &(N_C)^2(Tr[(\Lambda)^2])^2
\end{aligned} \tag{4.13}
$$

with N_C representing the number of quark colours. We also label the Feynman propagators for a quark as

$$
\overbrace{\bar{q}^A_{i\alpha}(x) q^B_{j\beta}(y)} = -i\delta_{\alpha\beta}\delta_{AB}S^A_{ji}(x-y)\,. \tag{4.14}
$$

We now transform the quark propagator so we can work in momentum space

$$
S^A_{ij}(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} S^A_{ij}(p) \tag{4.15}
$$

and substitute into (4.7) to obtain

$$
\begin{aligned}
\Pi_1(q^2) = &i(\gamma_5)_{ij}(\gamma_5)_{kl}(\gamma_5)_{mn}(\gamma_5)_{op}\int d^4x e^{i(q-p_1-p_2-p_3-p_4)\cdot x} \\
&\times \Bigg[\Bigg(f_1 \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{d^4p_4}{(2\pi)^4} S^A_{ni}(p_1)S^B_{jm}(p_2)S^C_{pk}(p_3)S^D_{lo}(p_4)\Bigg) \\
&+ \Bigg(f_2 \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{d^4p_4}{(2\pi)^4} S^A_{ni}(p_1)S^B_{jo}(p_2)S^C_{pk}(p_3)S^D_{lm}(p_4)\Bigg) \\
&+ \Bigg(f_3 \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{d^4p_4}{(2\pi)^4} S^A_{pi}(p_1)S^B_{jm}(p_2)S^C_{nk}(p_3)S^D_{lo}(p_4)\Bigg) \\
&+ \Bigg(f_4 \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{d^4p_4}{(2\pi)^4} S^A_{pi}(p_1)S^B_{jo}(p_2)S^C_{nk}(p_3)S^D_{lm}(p_4)\Bigg)\Bigg]\,.
\end{aligned} \tag{4.16}
$$

Using the definition of the Dirac delta function

$$
\delta(x'-x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x'-x)}\,, \tag{4.17}
$$

and distributing the γ_5's we obtain

$$
\begin{aligned}
\Pi_1(q^2) =& i \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} \\
& \times \Bigg[f_1(\gamma_5)_{mn} S^A_{ni}(p_1)(\gamma_5)_{ij} S^B_{jm}(p_2)(\gamma_5)_{op} S^C_{pk}(p_3)(\gamma_5)_{kl} S^D_{lo}(p_1+p_2+p_3-q) \\
& + f_2(\gamma_5)_{mn} S^A_{ni}(p_1)(\gamma_5)_{ij} S^B_{jo}(p_2)(\gamma_5)_{op} S^C_{pk}(p_3)(\gamma_5)_{kl} S^D_{lm}(p_1+p_2+p_3-q) \\
& + f_3(\gamma_5)_{op} S^A_{pi}(p_1)(\gamma_5)_{ij} S^B_{jm}(p_2)(\gamma_5)_{mn} S^C_{nk}(p_3)(\gamma_5)_{kl} S^D_{lo}(p_1+p_2+p_3-q) \\
& + f_4(\gamma_5)_{op} S^A_{pi}(p_1)(\gamma_5)_{ij} S^B_{jo}(p_2)(\gamma_5)_{mn} S^C_{nk}(p_3)(\gamma_5)_{kl} S^D_{lm}(p_1+p_2+p_3-q) \Bigg] .
\end{aligned}
\tag{4.18}
$$

Converting the above into traces results in

$$
\begin{aligned}
\Pi_1(q^2) =& i \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} \\
& + \Bigg[(2f_1) Tr[\gamma_5 S^A(p_1)\gamma_5 S^B(p_2)] Tr[\gamma_5 S^C(p_3)\gamma_5 S^D(p_1+p_2+p_3-q)] \\
& + (2f_2) Tr[\gamma_5 S^A(p_1)\gamma_5 S^B(p2)\gamma_5 S^C(p_3)\gamma_5 S^D(p_1+p_2+p_3-q)] \Bigg] .
\end{aligned}
\tag{4.19}
$$

If we substitute the trace identities obtained from (A.16) and (A.17) we get

$$
\begin{aligned}
\Pi_1(q^2) =& i \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} \\
& \times \frac{1}{(p_1^2+i\eta)(p_2^2+i\eta)(p_3^2+i\eta)[(p_1+p_2+p_3-q)^2+i\eta]} \\
& \times \Bigg[\Bigg(16(2f_1)(p_1\cdot p_2)(p_3\cdot(p_1+p_2+p_3-q)) \Bigg) + 4(2f_2) \Bigg((p_1\cdot p_2)(p_3\cdot(p_1+p_2+p_3-q)) \\
& + (p_1\cdot(p_1+p_2+p_3-q)(p_2\cdot p_3)) - (p_1\cdot p_3)(p_2\cdot(p_1+p_2+p_3-q)) \Bigg) \Bigg] .
\end{aligned}
\tag{4.20}
$$

Without loss of generality we can change the integration variables of the last three integrals in (4.20) and observe that these pieces are equal to each other. This simplifies our correlation function.

$$
\begin{aligned}
\Pi_1(q^2) =& i \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} \\
& \times \frac{(p_1\cdot p_2)(p_3\cdot(p_1+p_2+p_3-q))}{(p_1^2+i\eta)(p_2^2+i\eta)(p_3^2+i\eta)[(p_1+p_2+p_3-q)^2+i\eta]} \\
& \times \Bigg[16(2f_1) + 4(2f_2) \Bigg]
\end{aligned}
\tag{4.21}
$$

If we utilize the identity

$$
(p+q)^2 = p^2 + q^2 + 2p\cdot q \,,
\tag{4.22}
$$

and generalize to dimension $D = 4 + 2\epsilon$, we obtain the following form for the correlation function:

$$
\begin{aligned}
\Pi_1(q^2) =& i\left[16(2f_1) + 4(2f_2)\right] \\
& \times \frac{1}{2}\frac{1}{\nu^{2\epsilon}}\int \frac{d^D p_1}{(2\pi)^D}\frac{p_{1\mu}}{(p_1^2+i\eta)}\frac{1}{\nu^{2\epsilon}}\int \frac{d^D p_2}{(2\pi)^D}\frac{p_2^\mu}{(p_2^2+i\eta)} \\
& \times \frac{1}{\nu^{2\epsilon}}\int \frac{d^D p_3}{(2\pi)^D}\frac{\left[(p_3+(p_1+p_2-q))^2 - p_3^2 - (p_1+p_2-q)^2\right]}{(p_3^2+i\eta)[(p_3+p_1+p2-q)^2+i\eta]}\,.
\end{aligned}
\tag{4.23}
$$

This form of $\Pi(q^2)$ can now be solved using the Feynman integral formulas (3.50), (3.51) to obtain the result for the first order piece of the calculation

$$
\Pi_1(q^2) = \frac{1}{(4\pi)^6}\left(-\frac{1}{4\pi\nu^2}\right)^{3\epsilon} G_1(1,1)G_2(1,-1-\epsilon)\left[8f_1+2f_2\right]\left[G_1(1,-2\epsilon-3)-G_1(1,-2\epsilon-2)\right](q^2)^{4+3\epsilon}\,,
\tag{4.24}
$$

where $G_1(r,s)$ and $G_2(r,s)$ are defined in (3.50) and (3.51).

4.2 Next Highest Order Pieces in the Calculation of Correlation Function

Now that we have calculated the lowest order piece of our correlation function $\Pi(q^2)$ it is time to focus on the terms from (4.6) which have a gluon-quark interaction component. Let us look more closely at

$$
\Pi_2(q^2) = i\frac{g^2}{8}\int d^4x e^{iq\cdot x}\int d^4w\int d^4z \quad \Omega\,,
\tag{4.25}
$$

where Ω denotes the integrand in (4.6). Eventually $\Pi_2(q^2)$ can be split into two pieces; the first corresponding to terms which contain a quark self-energy piece and, the second, containing gluon interactions between the different quarks. First we must apply Wick's theorem to (4.6) and in doing so (4.6) will require some work. Let us focus on the quantity Ω, deconstructing it into its components:

$$
\begin{aligned}
\Omega =& \langle 0|T(J(x)\bar{q}(w)\lambda^a\gamma^\mu A^a_\mu(w)q(w)\bar{q}(z)\lambda^b\gamma^\nu A^b_\nu(z)q(z)J(0))|0\rangle \\
=& (\Lambda)_{AB}(\Lambda)_{CD}(\Lambda)_{EF}(\Lambda)_{GH}\langle 0|T\Bigg((\bar{q}^A_{i\alpha}(x)\gamma_5^{ij}q^B_{j\alpha}(x)\bar{q}^C_{k\beta}(x)\gamma_5^{kl}q^D_{l\beta}(x))\bar{q}^I_{m\gamma}(w)A^a_\mu(w)\gamma^\mu\lambda^a_{\gamma\psi}q^I_{n\psi}(w) \\
& \times \bar{q}^K_{o\xi}(z)A^b_\nu(z)\gamma^\nu\lambda^b_{\xi\sigma}q^K_{p\sigma}(z)(\bar{q}^E_{q\theta}(0)\gamma_5^{qr}q^F_{r\theta}(0)\bar{q}^G_{s\phi}(0)\gamma_5^{st}q^H_{t\phi}(0))\Bigg)|0\rangle\,.
\end{aligned}
\tag{4.26}
$$

Several things need to be taken into account when applying Wick's theorem to (4.26). First, we must notice that any term which involves a contraction between $A_\mu(w)$ and a quark field is zero (ie. $\langle 0|T(q(x)A_\mu(w))|0\rangle = 0$). Second, all terms that contain a normal ordered product of uncontracted fields will vanish since $\langle 0|N(\text{any operator})|0\rangle = 0$. Third, there are several terms corresponding to Figure 4.1 that will vanish since the Kronecker deltas conspire to give us traces of single λ^a matrices ($Tr[\lambda^a] = 0$). Finally, several of these terms are equal to each other allowing for the simplification

$$\Omega = i2f_6[T_1 + T_2 + T_3]\,, \tag{4.27}$$

where

$$T_1 \equiv -Tr[S(x)\gamma_5 S(-z)\gamma_\nu S(z-w)\gamma_\mu S(w-x)\gamma_5]Tr[S(x)\gamma_5 S(-x)\gamma_5]D^{\mu\nu}(w-z) \tag{4.28}$$

$$T_2 \equiv -Tr[S(-x)\gamma_5 S(x-w)\gamma_\mu S(w-z)\gamma_\nu S(z)\gamma_5]Tr[S(x)\gamma_5 S(-x)\gamma_5]D^{\mu\nu}(w-z) \tag{4.29}$$

$$T_3 \equiv Tr[S(w-x)\gamma_5 S(x-z)\gamma_\nu S(z)\gamma_5 S(-w)\gamma_\mu]Tr[S(x)\gamma_5 S(-x)\gamma_5]D^{\mu\nu}(w-z) \tag{4.30}$$

$$f_6 \equiv N_C Tr[\lambda^a\lambda^a](Tr[(\Lambda)^2])^2\,. \tag{4.31}$$

In order to make the connection between these terms and their corresponding Feynman diagrams we must examine the integrals containing these terms. Let us first have a look at the integral containing T_1:

$$\frac{ig^2}{4} f_6 \int d^4x e^{iq\cdot x} \int d^4w \int d^4z T_1\,. \tag{4.32}$$

After a Fourier transformation into momentum-space we obtain

$$\begin{aligned}
&-\frac{ig^2}{4} f_6 \int \cdots \int d^4x\, d^4w\, d^4z \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{d^4p_4}{(2\pi)^4}\frac{d^4p_5}{(2\pi)^4}\frac{d^4p_6}{(2\pi)^4}\frac{d^4k}{(2\pi)^4} \\
&\quad\times e^{ix\cdot(q-p_1+p_4-p_5+p_6)} e^{iw\cdot(p_3-p_4-k)} e^{iz\cdot(p_2+k-p_3)} D^{\mu\nu}(k) Tr[S(p_1)\gamma_5 S(p_2)\gamma_\nu S(p_3)\gamma_\mu S(p_4)\gamma_5] \\
&\quad\quad\times Tr[S(p_5)\gamma_5 S(p_6)\gamma_5]\,.
\end{aligned} \tag{4.33}$$

Let us use the definition of the Dirac delta function in (4.17) and take the p_2, p_4 and p_6 integrals

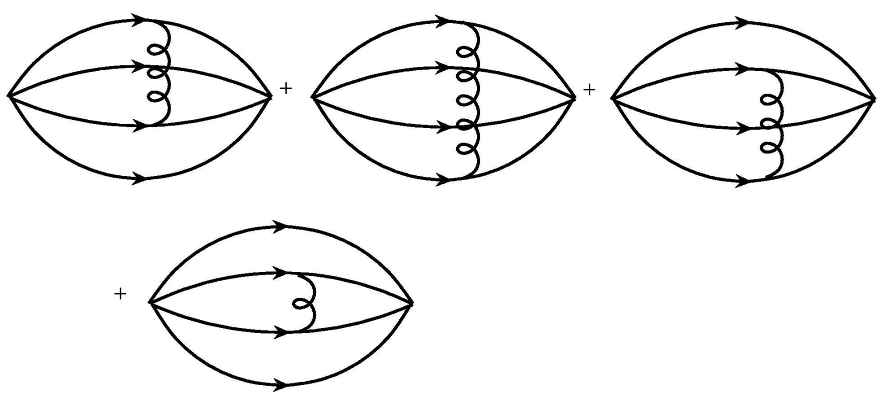

Figure 4.1: Feynman diagram for gluon exchange terms that vanish

of (4.33). We must also insert the value for the gluon propagator $D^{\mu\nu}(k)$ with $a = 1$ (Feynman gauge); the integral becomes

$$\frac{ig^2}{4} f_6 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} \frac{d^4p_6}{(2\pi)^4} Tr[S(p_1)\gamma_5 S(p_3-k)\gamma^\mu S(p_3)\gamma_\mu S(p_3-k)\gamma_5] \\ \times Tr[S(q-p_1+p_3-k+p_6)\gamma_5 S(p_6)\gamma_5]\,, \tag{4.34}$$

which corresponds to the diagram Figure 4.2.

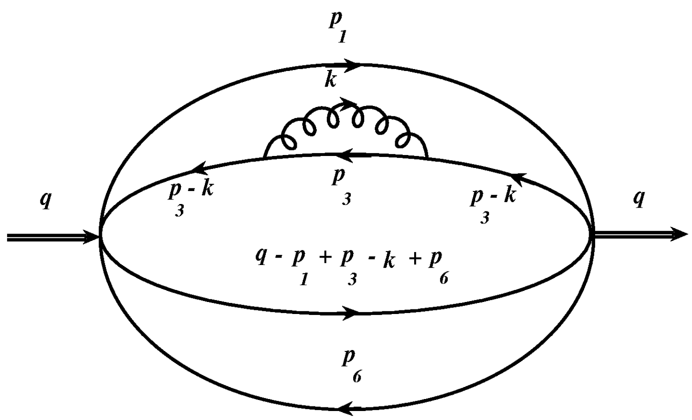

Figure 4.2: Feynman diagram for T_1

Applying the above techniques to T_2 reveals that T_1 and T_2 are equal to each other. This comes as no surprise since both these terms correspond to Feynman diagrams which contain quark propagators with self-energy pieces. That the calculation of the four terms corresponding to Figure 4.3 reduces to the calculation of one term containing a quark self-energy should also be evident since the diagrams would appear to be equivalent up to a change in variables. Once this term has been calculated we can analyze the remaining terms in the next highest order piece of our correlation function. Finally we will calculate the terms in $\Pi(q^2)$ which correspond to the diagrams in Figure 4.4. Once this process is complete we can combine all the terms and expand as $\epsilon \to 0$ to obtain the desired result for $\Pi(q^2)$.

4.3 Pieces of $\Pi(q^2)$ with a Quark Self-Energy Term

When we look at the pieces of $\Pi(q^2)$ that correspond to Figure 4.3 it is obvious that with a change of variables these diagrams can be made equivalent to one another. So without loss of generality the calculations of these pieces of $\Pi(q^2)$ can be reduced to the calculation of just one of these terms.

We will consider this term to be similar to the lowest order piece of $\Pi(q^2)$ with the exception that the quark propagator $S^A(p_1)$ must be modified to include the self energy term. This modified propagator shall be denoted as $S'^A(p_1)$. This simplification allows us to insert $S'^A(p_1)$ in place of

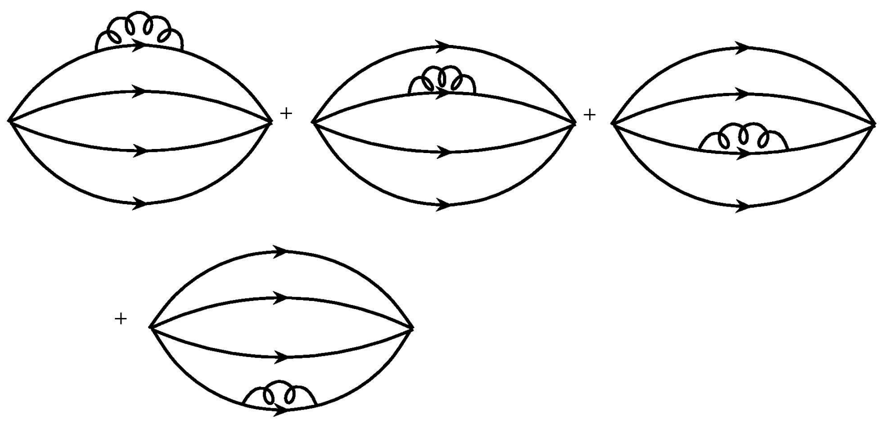

Figure 4.3: Feynman diagram for gluon self-energy pieces

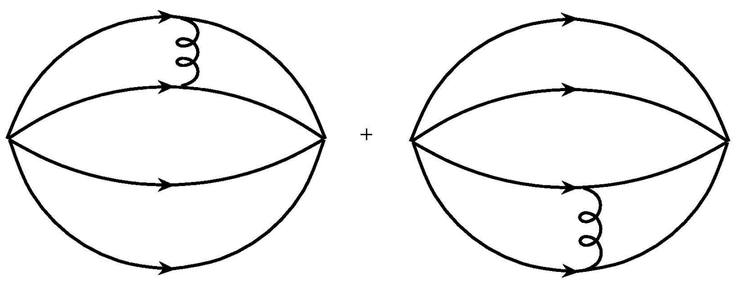

Figure 4.4: Feynman diagram for non-vanishing gluon exchange pieces

$S^A(p_1)$ in (4.19) and from there carry on the calculation exactly as was done for the lowest order piece of $\Pi(q^2)$. It is then necessary to determine the form of $S'^A(p_1)$.

We express $S'^A(p_1)$ as

$$iS'^A(p_1) = i\frac{\not{p}_1}{p_1^2}\left(-i\Sigma(\not{p}_1)\right)i\frac{\not{p}_1}{p_1^2}, \tag{4.35}$$

where $-i\Sigma(\not{p}_1)$ is the one-loop contribution to the quark self-energy. The quantity $-i\Sigma(\not{p}_1)$ is

$$\begin{aligned}-i\Sigma_{\beta\alpha}(\not{p}_1) &= \int \frac{d^4k}{(2\pi)^4}\left(i\frac{g\lambda^a_{\alpha\omega}\gamma^\mu}{2}\right)\left(i\frac{\not{p}_1+\not{k}}{(p_1+k)^2}\right)\left(i\frac{g\lambda^b_{\omega\beta}\gamma^\nu}{2}\right)\left[i\delta^{ab}\left(-g_{\mu\nu}+\frac{(1-a)k_\mu k_\nu}{k^2}\right)\frac{1}{k^2}\right] \\ &= -\frac{g^2}{4}[\lambda^a\lambda^a]_{\alpha\beta}\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^\mu(\not{p}_1+\not{k})\gamma^\nu}{(p_1+k)^2}\left[g_{\mu\nu}-\frac{(1-a)k_\mu k_\nu}{k^2}\right]\frac{1}{k^2}\,,\end{aligned} \tag{4.36}$$

where a is the gauge parameter and will be set $a = 1$ (Feynman gauge). If we use the identity of

Eq. (A.19) in [31]

$$(\lambda^a)_{\alpha\omega}(\lambda^a)_{\omega\beta} = 2\frac{N^2-1}{N}\delta_{\alpha\beta} \equiv 4C_2(R)\delta_{\alpha\beta}\,, \tag{4.37}$$

with $N = 3$, we obtain

$$-i\Sigma_{\beta\alpha}(\not{p}_1) = -\delta_{\beta\alpha}g^2\frac{C_2(R)}{\nu^{2\epsilon}}\int\frac{d^Dk}{(2\pi)^D}\frac{\gamma^\mu(\not{k}+\not{p}_1)\gamma_\mu}{[(k+p_1)^2+i\eta](k^2+i\eta)}\,. \tag{4.38}$$

Before we can arrange this integral in such a way as to be able to use the Feynman integral formulas from (3.50) and (3.51) we must deal with terms of the form

$$\gamma^\mu\not{k}\gamma_\mu\,, \tag{4.39}$$

by using the anticommutation relationship

$$\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,. \tag{4.40}$$

Applying this to (4.39) we obtain

$$\begin{aligned}\gamma^\mu\not{k}\gamma_\mu &= \gamma^\mu(\gamma^\nu k_\nu)\gamma_\mu\\ &= (2g^{\mu\nu}-\gamma^\nu\gamma^\mu)k_\nu\gamma_\mu\\ &= \not{k}(2-\gamma^\mu\gamma_\mu)\\ &= \not{k}(2-D) = -2\not{k}(1+\epsilon)\,.\end{aligned} \tag{4.41}$$

Using the result from (4.41) we can rearrange (4.38) in terms of Feynman integrals

$$\begin{aligned}i\Sigma_{\beta\alpha}(p_1) =& \delta_{\beta\alpha}g^2C_2(R)(2+2\epsilon)\left(\frac{\gamma_\nu}{\nu^{2\epsilon}}\int\frac{d^Dk}{(2\pi)^D}\frac{k^\nu}{[(k+p_1)^2+i\eta](k^2+i\eta)}\right.\\ &\left.+\frac{\not{p}_1}{\nu^{2\epsilon}}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{[(k+p_1)^2+i\eta](k^2+i\eta)}\right)\,.\end{aligned} \tag{4.42}$$

Now we can apply the Feynman integral formulas (3.50), (3.51) and (3.52) we can solve this expression to obtain

$$i\Sigma_{\beta\alpha}(p_1) = i\delta_{\beta\alpha}g^2\frac{C_2(R)}{(4\pi)^2}\left(-\frac{p_1^2}{4\pi\nu^2}\right)^\epsilon(2+2\epsilon)\not{p}_1[G_2(1,1)+G_1(1,1)]\,. \tag{4.43}$$

Substituting this result into (4.35) leads to

$$\begin{aligned}iS'^A(p_1) =& i\frac{\not{p}_1}{p_1^2}\left(i\delta_{\beta\alpha}g^2\frac{C_2(R)}{(4\pi)^2}\left(-\frac{p_1^2}{4\pi\nu^2}\right)^\epsilon(2+2\epsilon)\not{p}_1[G_2(1,1)+G_1(1,1)]\right)i\frac{\not{p}_1}{p_1^2}\\ =& -i\delta_{\beta\alpha}\left(\frac{g}{4\pi}\right)^2C_2(R)\left(-\frac{1}{4\pi\nu^2}\right)^\epsilon p_1^{2\epsilon-2}(2+2\epsilon)\not{p}_1[G_2(1,1)+G_1(1,1)]\\ =& K\delta_{\beta\alpha}\frac{i}{(4\pi)^2}\left(-\frac{1}{4\pi\nu^2}\right)^\epsilon p_1^{2\epsilon-2}\not{p}_1\,,\end{aligned} \tag{4.44}$$

where K is

$$K \equiv -g^2 C_2(R)(2+2\epsilon)[G_2(1,1)+G_1(1,1)]\,. \tag{4.45}$$

Now that we have a result for our modified propagator $S'^A(p_1)$ we can apply it to the problem of solving the next highest order pieces of the correlation function that correspond to the diagrams terms with a self energy piece. Substituting the modified propagator $S'^A(p_1)$ into (4.19) we can now continue with our calculation of the correlation function piece corresponding to Figure 4.5.

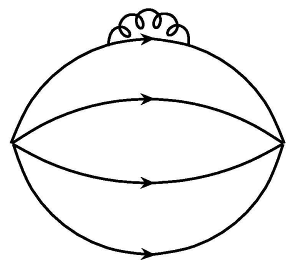

Figure 4.5: Feynman diagram for gluon self-energy pieces

The self-energy term is

$$\begin{aligned}\Pi(q^2)_{SE} =& i\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\\ &+\Bigg[(2f_1)Tr[\gamma_5 S'^A(p_1)\gamma_5 S^B(p_2)]Tr[\gamma_5 S^C(p_3)\gamma_5 S^D(p_1+p_2+p_3-q)]\\ &+(2f_2)Tr[\gamma_5 S'^A(p_1)\gamma_5 S^B(p2)\gamma_5 S^C(p_3)\gamma_5 S^D(p_1+p_2+p_3-q)]\Bigg]\,.\end{aligned} \tag{4.46}$$

After substituting the values for the traces in (A.16) and (A.17), we can separate this integral into two separate integrals

$$\Pi(q^2)_{SE} = i\Bigg[2f_1 I_I(p_1,p_2,p_3,q)+2f_2 I_{II}(p_1,p_2,p_3,q)\Bigg]\,, \tag{4.47}$$

where

$$I_I(p_1,p_2,p_3,q) = \frac{16K}{\nu^{6\epsilon}(4\pi)^2}\left(-\frac{1}{4\pi\nu^2}\right)^{\epsilon}\int\frac{d^Dp_1}{(2\pi)^D}\int\frac{d^Dp_2}{(2\pi)^D}\int\frac{d^Dp_3}{(2\pi)^D}\frac{p_1\cdot p_2[p_3\cdot(p_3-l)]}{(p_1^2)^{1-\epsilon}(p_2^2)(p_3)^2(p_3-l)^2} \tag{4.48}$$

$$\begin{aligned}I_{II}(p_1,p_2,p_3,q) =& 4\frac{K}{(4\pi)^2}\left(-\frac{1}{4\pi\nu^2}\right)^{\epsilon}\frac{1}{\nu^{6\epsilon}}\int\frac{d^Dp_1}{(2\pi)^D}\int\frac{d^Dp_2}{(2\pi)^D}\int\frac{d^Dp_3}{(2\pi)^D}\\ &\times\left[\frac{(p_1\cdot p_2)(p_3\cdot k)+(p_1\cdot k)(p_2\cdot p_3)-(p_1\cdot p_3)(p_2\cdot k)}{(p_1^2)^{1-\epsilon}(p_2^2)(p_3^2)(k^2)}\right],\end{aligned} \tag{4.49}$$

and where we have used the substitutions $l=-p_1-p_2+q$ and $k=p_1+p_2+p_3-q$. Let us first look at $I_I(p_1,p_2,p_3,q)$. After a little work, and removing the tadpole integrals (one-point functions corresponding to Feynman diagrams with one external leg), we can rearrange (4.48) into

$$
\begin{aligned}
I_I(p_1,p_2,p_3,q) =& -8\frac{K}{(4\pi)^2}\left(-\frac{1}{4\pi\nu^2}\right)^{\epsilon}\frac{1}{\nu^{2\epsilon}}\int\frac{d^Dp_1}{(2\pi)^D}\frac{p_{1\mu}}{(p_1^2)^{1-\epsilon}}\frac{1}{\nu^{2\epsilon}}\int\frac{d^Dp_2}{(2\pi)^D}\frac{p_2^{\mu}}{(p_2^2)}\frac{l^2}{\nu^{2\epsilon}} \\
&\times\int\frac{d^Dp_3}{(2\pi)^D}\frac{1}{(p_3^2)(p_3-l)^2}\,.
\end{aligned}
\tag{4.50}
$$

Eq. (4.50) is now in a form where we can apply the Feynman integral formulas and solve for $I_I(p_1,p_2,p_3,q)$ using similar techniques as before. We obtain the result

$$
\begin{aligned}
I_I(p_1,p_2,p_3,q) =& -\frac{i4K}{(4\pi)^8}\left(-\frac{1}{4\pi\nu^2}\right)^{4\epsilon}G_1(1,1)G_2(1,-1-\epsilon) \\
&\times\Bigg[G_1(-\epsilon,-2-2\epsilon)-G_1(1-\epsilon,-2-2\epsilon)+G_1(1-\epsilon,-3-2\epsilon)\Bigg](q^2)^{4\epsilon+4} \\
=& -\frac{i4K'}{(4\pi)^8}\left(-\frac{1}{4\pi\nu^2}\right)^{4\epsilon}(q^2)^{4\epsilon+4}\,,
\end{aligned}
\tag{4.51}
$$

where

$$
K' = KG_1(1,1)G_2(1,-1-\epsilon)\Bigg[G_1(-\epsilon,-2-2\epsilon)-G_1(1-\epsilon,-2-2\epsilon)+G_1(1-\epsilon,-3-2\epsilon)\Bigg]. \tag{4.52}
$$

Now we can examine the second piece of (4.47), $I_{II}(p_1,p_2,p_3,q)$, and reduce it to simple one-loop Feynman Integrals. Without loss of generality we can exchange the indices $p_2 \to p_3$ in the first integral of $I_{II}(p_1,p_2,p_3,q)$ to obtain

$$
\begin{aligned}
I_{II}(p_1,p_2,p_3,q) =& 4\frac{K}{(4\pi)^2}\left(-\frac{1}{4\pi\nu^2}\right)^{\epsilon}\frac{1}{\nu^{6\epsilon}}\int\frac{d^Dp_1}{(2\pi)^D}\int\frac{d^Dp_2}{(2\pi)^D}\int\frac{d^Dp_3}{(2\pi)^D} \\
&\times\left[\frac{(p_1\cdot p_3)(p_2\cdot k)+(p_1\cdot k)(p_2\cdot p_3)-(p_1\cdot p_3)(p_2\cdot k)}{(p_1^2)^{1-\epsilon}(p_2^2)(p_3^2)(k^2)}\right] \\
=& 4\frac{K}{(4\pi)^2}\left(-\frac{1}{4\pi\nu^2}\right)^{\epsilon}\frac{1}{\nu^{6\epsilon}}\int\frac{d^Dp_1}{(2\pi)^D}\int\frac{d^Dp_2}{(2\pi)^D}\int\frac{d^Dp_3}{(2\pi)^D} \\
&\times\left[\frac{(p_1\cdot k)(p_2\cdot p_3)}{(p_1^2)^{1-\epsilon}(p_2^2)(p_3^2)(k^2)}\right].
\end{aligned}
\tag{4.53}
$$

Substitute $p_1 - w = k$, where $w = -p_2 - p_3 + q$, into (4.53) and rearrange $I_{II}(p_1,p_2,p_3,q)$ into a form ready for the application of the Feynman integral formulae as before

$$
\begin{aligned}
I_{II}(p_1,p_2,p_3,q) =& 4\frac{K}{(4\pi)^2}\left(-\frac{1}{4\pi\nu^2}\right)^{\epsilon}\frac{1}{\nu^{2\epsilon}}\int\frac{d^Dp_2}{(2\pi)^D}\frac{p_{2\mu}}{(p_2)^2}\frac{1}{\nu^{2\epsilon}}\int\frac{d^Dp_3}{(2\pi)^D}\frac{p_3^{\mu}}{(p_3^2)}\frac{1}{\nu^{2\epsilon}} \\
&\times\int\frac{d^Dp_1}{(2\pi)^D}\frac{p_1\cdot(p_1-w)}{(p_1^2)^{1-\epsilon}(p_1-w)^2}\,.
\end{aligned}
\tag{4.54}
$$

Once again we use similar techniques as before to solve this integral. We obtain the result

$$
\begin{aligned}
I_{II}(p_1,p_2,p_3,q) &= \frac{iK}{(4\pi)^8}\left(-\frac{1}{4\pi\nu^2}\right)^{4\epsilon} G_2(1,-2\epsilon-1)\Bigg[G_1(-\epsilon,1)-G_1(1-\epsilon,1)\Bigg] \\
&\quad\times\Bigg[G_1(1,-3\epsilon-3)-G_1(1,-3\epsilon-2)\Bigg](q^2)^{4\epsilon+4} \\
&= \frac{i\tilde{K}}{(4\pi)^8}\left(-\frac{1}{4\pi\nu^2}\right)^{4\epsilon}(q^2)^{4\epsilon+4}\,,
\end{aligned}
\tag{4.55}
$$

where

$$
\tilde{K} \equiv KG_2(1,-2\epsilon-1)\Bigg[G_1(-\epsilon,1)-G_1(1-\epsilon,1)\Bigg]\Bigg[G_1(1,-3\epsilon-3)-G_1(1,-3\epsilon-2)\Bigg]. \tag{4.56}
$$

Combining this with (4.51) we have the analytic result for the piece of $\Pi_{SE}(q^2)$ corresponding to Figure 4.5

$$
\Pi_{SE}(q^2) = \frac{1}{(4\pi)^8}\left(\frac{1}{4\pi\nu^2}\right)^{4\epsilon}(-q^2)^{4\epsilon+4}\Bigg[-(2f_2)\tilde{K}+4(2f_1)K'\Bigg]. \tag{4.57}
$$

4.4 Piece of $\Pi(q^2)$ with gluon exchange between quarks

The remaining piece of the calculation of our correlation function $\Pi(q^2)$ contains terms which correspond to the first term in Figure 4.4. In Section 4.2 we saw that only six terms remained after the application of Wick's theorem. The first four that corresponded to Feynman diagrams with a quark self-energy piece reduced to the calculation of one Feynman integral and have been dealt with in the previous section. The remaining two terms correspond to Feynman diagrams in which there is a gluon exchange between two quarks. Before we jump into the calculation of these terms it is useful to examine terms that vanished in Section 4.2 as a result of containing the traces of single λ^a matrices. It is interesting to note that of the six terms that involved a gluon exchange only the two terms in which the exchange was between a quark-antiquark pair survived. In order to eliminate the terms that are equal to zero it is sufficient to look at the contractions resulting from the application of Wick's theorem. For example let us have a look at the term corresponding to the second term in Figure 4.1.

$$
\begin{aligned}
&(\Lambda)_{AB}(\Lambda)_{CD}(\Lambda)_{EF}(\Lambda)_{GH}\gamma_5^{ij}\gamma_5^{kl}\gamma_5^{qr}\gamma_5^{st}\gamma_\mu^{mn}\gamma_\nu^{op}\lambda^a_{\gamma\psi}\lambda^b_{\xi\sigma}\langle 0|T(A^{\mu\nu}(w)A^{\mu\nu}(z))|0\rangle\langle 0|T(\bar{q}_{i\alpha}(x)q_{n\psi}(w))|0\rangle \\
&\qquad\times\langle 0|T(q_{j\alpha}(x)\bar{q}_{q\theta}(0))|0\rangle\langle 0|T(\bar{q}_{k\beta}(x)q_{t\phi}(0))|0\rangle\langle 0|T(q_{l\beta}(x)\bar{q}_{o\xi}(z))|0\rangle \\
&\qquad\times\langle 0|T(\bar{q}_{m\gamma}(w)q_{r\theta}(0))|0\rangle\langle 0|T(q_{p\sigma}(z)\bar{q}_{s\phi}(0))|0\rangle \\
&= i(\Lambda)_{AB}(\Lambda)_{CD}(\Lambda)_{EF}(\Lambda)_{GH}\lambda^a_{\gamma\psi}\lambda^b_{\xi\sigma}\delta_{ab}\delta_{\alpha\psi}\delta_{\gamma\theta}\delta_{\sigma\phi}\delta_{\beta\xi}\delta_{\beta\phi}\delta_{\alpha\theta}D^{\mu\nu}(w-z)Tr[S(x-w)\gamma_5S(x)\gamma_5S(w)\gamma_\mu] \\
&\qquad\times Tr[S(x)\gamma_5S(x-z)\gamma_\nu S(z)\gamma_5]
\end{aligned}
\tag{4.58}
$$

It is not necessary to complete the entire calculation of this term. A simple analysis of the Kronecker delta (δ_{ij}) combination and the lambda matrices arising from the contractions returns the result

$$
\begin{aligned}
&\lambda^a_{\gamma\psi}\lambda^b_{\xi\sigma}\delta_{ab}\delta_{\alpha\psi}\delta_{\gamma\theta}\delta_{\sigma\phi}\delta_{\beta\xi}\delta_{\beta\phi}\delta_{\alpha\theta} \\
&= \lambda^a_{\gamma\psi}\lambda^a_{\xi\sigma}\delta_{\psi\theta}\delta_{\xi\phi}\delta_{\gamma\theta}\delta_{\sigma\phi} = \lambda^a_{\gamma\psi}\lambda^a_{\xi\sigma}\delta_{\gamma\psi}\delta_{\sigma\xi} = Tr[\lambda^a]Tr[\lambda^a]\,.
\end{aligned}
\tag{4.59}
$$

Since the λ^a matrices are traceless, this term vanishes. Applying this same analysis to the terms corresponding to the terms in Figure 4.1 reveals that all these terms vanish. The only non-zero gluon-exchange terms are the ones in Figure 4.4. The calculation of the remaining pieces of $\Pi(q^2)$ has been reduced to the calculation of the Feynman integral containing the term T_3 from (4.30):

$$
\begin{aligned}
I_{T_3} &\equiv -f_6\frac{g^2}{8}\int d^4x e^{iq\cdot x}\int d^4z T_3 \\
&= -f_6\frac{g^2}{8}\int d^4x e^{iq\cdot x}\int d^4z Tr[S(w-x)\gamma_5 S(x-z)\gamma_\nu S(z)\gamma_5 S(-w)\gamma_\mu] \\
&\qquad\qquad \times Tr[S(x)\gamma_5 S(-x)\gamma_5]D^{\mu\nu}(w-z)\,.
\end{aligned}
\tag{4.60}
$$

Now after a Fourier transformation into momentum space we obtain

$$
\begin{aligned}
I_{T_3} =& -f_6\frac{g^2}{8}\int d^4x e^{iq\cdot x}\int d^4z\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_6}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4k^2} \\
&\times e^{-ip_1\cdot(w-x)}e^{-ip_2\cdot(x-z)}e^{-ip_3\cdot z}e^{-ip_4\cdot(-w)}e^{-ip_5\cdot x}e^{-ip_6\cdot(-x)}e^{-ik\cdot(w-z)} \\
&\times Tr[S(p_1)\gamma_5 S(p_2)\gamma_\nu S(p_3)\gamma_5 S(p_4)\gamma_\mu]Tr[S(p_5)\gamma_5 S(p_6)\gamma_5]D^{\mu\nu}(k) \\
=& -f_6\frac{g^2}{8}\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\int\frac{d^4p_6}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4k^2} \\
&\times \delta(q+p_1-p_3+k+p_6-p_5)\delta(p_4-(p_1+k))\delta(p_2-(p_3-k)) \\
&\times Tr[S(p_1)\gamma_5 S(p_2)\gamma^\mu S(p_3)\gamma_5 S(p_4)\gamma_\mu]Tr[S(p_5)\gamma_5 S(p_6)\gamma_5] \\
=& -f_6\frac{g^2}{8}\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\int\frac{d^4p_6}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4k^2}Tr[S(p_1)\gamma_5 S(p_3-k)\gamma^\mu S(p_3)\gamma_5 S(p_1+k)\gamma_\mu] \\
&\times Tr[S(q+p_1-p_3+k+p_6)\gamma_5 S(p_6)\gamma_5]\,.
\end{aligned}
\tag{4.61}
$$

4.5 Substitution of Variables in Order to Solve I_{T_3}

The final piece of our correlation function $\Pi(q^2)$ to be calculated is not as straightforward to calculate as the previous ones. Up until now the calculation of $\Pi(q^2)$ involved an application of Wick's theorem, a transformation into momentum space, the application of trace theorems and finally, the solution of these terms using Feynman's one-loop integral formulas. The integral (4.61) is a slightly different beast. It does not immediately, after the application of trace theorems, resemble any of the standard Feynman integrals we have to work with. The integral (4.61) contains terms that can not be separated and integrated one piece at a time, nor does it match any of the two and three-loop integral formulas that are known. While it is possible that with some time and effort an analytical solution could be hammered out for this integral, another shortcut presents itself. It turns out that a simple substitution of the variables transforms (4.61) into a form in which Feynman's integral formulas and the two and thee-loop formulas developed in Sections 1.1.13 and 1.1.14 can be applied directly.

First, without loss of generality we can substitute $p_1 \to -p_1$, $p_3 \to -p_3$ and $k \to -k$ to obtain

$$- f_6 \frac{g^2}{8} \int \frac{d^4 p_1}{(2\pi)^4} \int \frac{d^4 p_3}{(2\pi)^4} \int \frac{d^4 p_6}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4 k^2} \times Tr[S(p_1)\gamma_5 S(p_3 - k)\gamma^\mu S(p_3)\gamma_5 S(p_1 + k)\gamma_\mu] Tr[S(q - p_1 + p_3 - k + p_6)\gamma_5 S(p_6)\gamma_5] \,, \tag{4.62}$$

which corresponds to the Feynman diagram in Figure 4.6.

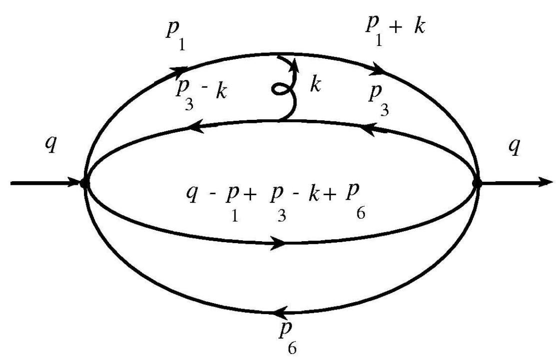

Figure 4.6: Feynman diagram for gluon exchange term before substitution

Following the above with the substitution

$$q = p, \quad p_1 = -k_1; \quad p_1 + k = -k_2; \quad k = k_1 - k_2; \quad p_3 - k = -(k_1 + \tilde{q}); \quad p_3 = -(k_2 + \tilde{q}); \quad p_6 = -(p - k_3); \quad q - p_1 + p_3 - k + p_6 = k_3 - \tilde{q} \,, \tag{4.63}$$

our integral becomes

$$\begin{aligned} I_{T_3} = &- f_6 \frac{g^2}{8} \int \frac{d^4 \tilde{q}}{(2\pi)^4} \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \int \frac{d^4 k_3}{(2\pi)^4 k^2} \\ &\times Tr[S(-k_1)\gamma_5 S(-k_1 - \tilde{q})\gamma^\mu S(-k_2 - \tilde{q})\gamma_5 S(-k_2)\gamma_\mu] Tr[S(k_3 - \tilde{q})\gamma_5 S(k_3 - p)\gamma_5] \\ = &-f_6 \frac{g^2}{8} \int \frac{d^4 \tilde{q}}{(2\pi)^4} \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \int \frac{d^4 k_3}{(2\pi)^4 k^2} \\ &\times Tr[S(k_1)\gamma_5 S(k_1 + \tilde{q})\gamma^\mu S(k_2 + \tilde{q})\gamma_5 S(k_2)\gamma_\mu] Tr[S(\tilde{q} - k_3)\gamma_5 S(p - k_3)\gamma_5] \,. \end{aligned} \tag{4.64}$$

This corresponds to Figure 4.7.

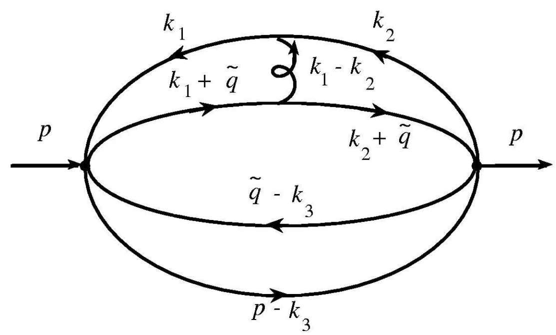

Figure 4.7: Feynman diagram for gluon exchange term after substitution

4.6 Calculation of I_{T_3}

Now that we have a workable form for I_{T_3}, we can insert the values for the propagators

$$\begin{aligned} I_{T_3} = &- f_6 \frac{g^2}{8} \int \frac{d^4\tilde{q}}{(2\pi)^4} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \\ &\times \frac{Tr[\not{k}_1\gamma_5(\not{k}_1 + \not{\tilde{q}})\gamma^\mu(\not{k}_2 + \not{\tilde{q}})\gamma_5 \not{k}_2\gamma_\mu]Tr[(\not{\tilde{q}} - \not{k}_3)\gamma_5(\not{p} - \not{k}_3)\gamma_5]}{k_1^2(k_1+\tilde{q})^2(k_2+\tilde{q})^2k_2^2(k_3-\tilde{q})^2(p-k_3)^2(k_1-k_2)^2}. \end{aligned} \tag{4.65}$$

As we have done before we shall apply the trace theorems from (A.16) and (A.18) to (4.64) in order to obtain

$$\begin{aligned} I_{T_3} =& f_6 2g^2 \int \frac{d^4\tilde{q}}{(2\pi)^4} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \\ &\times \left[\frac{(\tilde{q}-k_3)\cdot(p-k_3)}{k_1^2(k_1+\tilde{q})^2(k_2+\tilde{q})^2k_2^2(k_3-\tilde{q})^2(p-k_3)^2(k_1-k_2)^2} \right] \\ &\times \left[\frac{(4-D)}{4} \Big([(k_2+\tilde{q})\cdot(k_1+\tilde{q})][k_1\cdot k_2] - [(k_2+\tilde{q})\cdot k_1][k_2\cdot(k_1+\tilde{q})] \right. \\ &\left. + [k_1\cdot(k_1+\tilde{q})][k_2\cdot(k_2+\tilde{q})] \Big) - \Big([k_2\cdot(k_2+\tilde{q})][k_1\cdot(k_1+\tilde{q})] \Big) \right]. \end{aligned} \tag{4.66}$$

We can rearrange this integral as follows:

$$\begin{aligned} I_{T_3} =& f_6 2g^2 \int \frac{d^4\tilde{q}}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \frac{(\tilde{q}-k_3)\cdot(p-k_3)}{(k_3-\tilde{q})^2(p-k_3)^2} \\ &\times \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \frac{1}{k_1^2(k_1+\tilde{q})^2(k_2+\tilde{q})^2k_2^2(k_1-k_2)^2} \\ &\times \left[\frac{(4-D)}{4} \Big([(k_2+\tilde{q})\cdot(k_1+\tilde{q})][k_1\cdot k_2] - [(k_2+\tilde{q})\cdot k_1][k_2\cdot(k_1+\tilde{q})] \right. \\ &\left. + [k_1\cdot(k_1+\tilde{q})][k_2\cdot(k_2+\tilde{q})] \Big) - \Big([k_2\cdot(k_2+\tilde{q})][k_1\cdot(k_1+\tilde{q})] \Big) \right]. \end{aligned} \tag{4.67}$$

In order to solve this integral (I_{T_3}) we will want to break it up into pieces and calculate each one separately. First we will look at the integration over k_1 and k_2 and leave the k_3 and $\tilde{q}$ integration for later. As we did before, we can use the identities $a\cdot b = \frac{1}{2}[a^2+b^2-(a-b)^2]$ and $a\cdot b = \frac{1}{2}[(a+b)-a^2-b^2]$ to remove the dot products from the numerator. We can then expand the products in the numerator. After a little work we obtain the following for I_{T_3}:

$$I_{T_3} = \frac{f_6 g^2}{2}\left[\frac{(4-D)}{4}[4I^1_{T_3} + 2I^7_{T_3}] - I^2_{T_3} - 2I^1_{T_3} + 2I^3_{T_3} - I^6_{T_3} + 2I^5_{T_3} - I^4_{T_3}\right], \tag{4.68}$$

where

$$I^i_{T_3} = \frac{1}{\nu^{4\epsilon}}\int \frac{d^D\tilde{q}}{(2\pi)^D}\int \frac{d^Dk_3}{(2\pi)^D}\frac{(\tilde{q}-k_3)\cdot(p-k_3)}{(k_3-\tilde{q})^2(p-k_3)^2}\tilde{I}^i_{T_3} \qquad \text{for } 1\leq i\leq 7 \quad , \tag{4.69}$$

and

$$\tilde{I}^1_{T_3} = \frac{1}{\nu^{4\epsilon}}\int \frac{d^Dk_1}{(2\pi)^D}\int \frac{d^Dk_2}{(2\pi)^D}\frac{1}{k^2(k_1+\tilde{q})^2(k_1-k_2)^2} \tag{4.70}$$

$$\tilde{I}^2_{T_3} = \frac{1}{\nu^{4\epsilon}}\int \frac{d^Dk_1}{(2\pi)^D}\int \frac{d^Dk_2}{(2\pi)^D}\frac{1}{(k_1+\tilde{q})^2(k_2+\tilde{q})^2(k_1-k_2)} \tag{4.71}$$

$$\tilde{I}^3_{T_3} = \frac{\tilde{q}^2}{\nu^{4\epsilon}}\int \frac{d^Dk_1}{(2\pi)^D}\int \frac{d^Dk_2}{(2\pi)^D}\frac{1}{k_2^2(k_1+\tilde{q})^2(k_2+\tilde{q})^2(k_1-k_2)^2} \tag{4.72}$$

$$\tilde{I}^4_{T_3} = \frac{\tilde{q}^4}{\nu^{4\epsilon}}\int \frac{d^Dk_1}{(2\pi)^D}\int \frac{d^Dk_2}{(2\pi)^D}\frac{1}{k_1^2k_2^2(k_1+\tilde{q})^2(k_2+\tilde{q})^2(k_1-k_2)^2} \tag{4.73}$$

$$\tilde{I}^5_{T_3} = \frac{\tilde{q}^2}{\nu^{4\epsilon}}\int \frac{d^Dk_1}{(2\pi)^D}\int \frac{d^Dk_2}{(2\pi)^D}\frac{1}{k_1^2k_2^2(k_2+\tilde{q})^2(k_1-k_2)^2} \tag{4.74}$$

$$\tilde{I}^6_{T_3} = \frac{1}{\nu^{4\epsilon}}\int \frac{d^Dk_1}{(2\pi)^D}\int \frac{d^Dk_2}{(2\pi)^D}\frac{1}{k_1^2k_2^2(k_1-k_2)^2} \tag{4.75}$$

$$\tilde{I}^7_{T_3} = \frac{\tilde{q}^2}{\nu^{4\epsilon}}\int \frac{d^Dk_1}{(2\pi)^D}\int \frac{d^Dk_2}{(2\pi)^D}\frac{1}{k_1^2k_2^2(k_1^2k_2^2(k_1+\tilde{q})^2(k_2+\tilde{q})^2}\,. \tag{4.76}$$

The above $\tilde{I}^i_{T_3}$'s are all simple two-loop Feynman integrals that can be separated and solved by iteration except for $I^4_{T_3}$. The integral $I^4_{T_3}$ is equivalent to $I'_2(1,1,1,1,1)$ calculated in (3.86) and so we use this for the value of $I^4_{T_3}$. Combining all the values for these terms gives the result

$$\begin{aligned}
I_{T_3} =& -\frac{g^2}{2(4\pi)^4}f_6\left(-\frac{1}{4\pi\nu^2}\right)^{2\epsilon}G_1(1,1)\\
&\times\left[\frac{(4-D)}{4}[4G_1(-\epsilon,1)+2G_1(1-\epsilon,1)] - 2G_1(-\epsilon,1) + 2G_1(1,1-\epsilon) + 2G_1(1-\epsilon,1)\right.\\
&\left. - G_1(2,1) + G_1(2,1-\epsilon)\right]\frac{1}{\nu^{4\epsilon}}\int\frac{d^D\tilde{q}}{(2\pi)^D}\int\frac{d^Dk_3}{(2\pi)^D}\frac{(\tilde{q}-k_3)\cdot(p-k_3)}{(k_3-\tilde{q})^2(p-k_3)^2}\tilde{q}^{4\epsilon+2}\,.
\end{aligned} \tag{4.77}$$

The integral over $\tilde{q}$ and k_3 can now be easily solved using all the same methods as before to obtain our final result. We will also substitute $p=q$ back into I_{T_3}

$$\begin{aligned}
I_{T_3} =& -f_6\frac{1}{(4\pi)^8}g^2\left(-\frac{1}{4\pi\nu^2}\right)^{4\epsilon}(G_1(1,1))^2G_1(-2\epsilon-1,-\epsilon-1)\\
&\times\left[\frac{(4-D)}{4}[4G_1(-\epsilon,1)+2G_1(1-\epsilon,1)] - 2G_1(-\epsilon,1) + 2G_1(1,1-\epsilon) + 2G_1(1-\epsilon,1)\right.\\
&\left. - G_1(2,1) + G_1(2,1-\epsilon)\right](q^2)^{4\epsilon+4}\,.
\end{aligned} \tag{4.78}$$

Chapter 5

Results

Now that we have calculated all the pieces of the correlation function $\Pi(q^2)$ for a $q\bar{q}q\bar{q}$ state all that is left is to arrange the terms into the proper form. First let us look at the lowest order piece

$$\Pi_1(q^2) = \frac{8f_1 + 2f_2}{(4\pi)^6} q^8 \left(-\frac{q^2}{4\pi\nu^2}\right)^{3\epsilon} G_1(1,1) G_2(1,-1-\epsilon)[G_1(1,-2\epsilon-3) - G_1(1,-2\epsilon-2)] \,. \quad (5.1)$$

Now using the expansion for $\Gamma(z)$ in (A.27) we can expand the $G_1(r,s)$ and $G_2(r,s)$ terms in (5.1) keeping terms up to but not including $O(\epsilon)$ to obtain

$$\Pi_1(q^2) = \frac{8f_1 + 2f_2}{(4\pi)^6} q^8 \left(-\frac{q^2}{4\pi\nu^2}\right)^{3\epsilon} \left[\left(\frac{1}{2160}\right)\frac{1}{\epsilon} + \left(\frac{\gamma}{720} - \frac{683}{129600}\right)\right] . \quad (5.2)$$

Now we deal with the terms that contain powers of ϵ:

$$\begin{aligned} \left(\frac{-q^2}{\nu^2}\right)^{3\epsilon} &= \exp\left(\ln\left[\left(\frac{-q^2}{\nu^2}\right)^{3\epsilon}\right]\right) \\ &= 1 + 3\epsilon \ln\left[\left(\frac{-q^2}{\nu^2}\right)\right] + \frac{9\epsilon^2}{2}\ln^2\left[\left(\frac{-q^2}{\nu^2}\right)\right] . \end{aligned} \quad (5.3)$$

The same process is applied to the $\frac{1}{(4\pi)^{3\epsilon}}$ term and both are substituted back into (5.2). After distribution we obtain

$$\begin{aligned} \Pi_1(q^2) = &\frac{8f_1 + 2f_2}{(4\pi)^6} q^8 \left[\left(\frac{1}{2160}\right)\frac{1}{\epsilon} + \frac{\gamma}{720} - \frac{683}{129600} - \frac{1}{720}\ln(4\pi) \right. \\ &\left. + \frac{1}{720}\ln\left(-\frac{q^2}{\nu^2}\right) + O(\epsilon)\right] . \end{aligned} \quad (5.4)$$

After some rearranging and dropping the terms of order $O(\epsilon)$ we obtain

$$\Pi_1(q^2) = \frac{1}{(4\pi)^6} q^8 \left[\hat{B}\ln\left(-\frac{q^2}{\nu^2}\right) + \frac{\hat{E}}{\epsilon} + \hat{F}\right] , \quad (5.5)$$

where

$$\hat{B} \equiv \frac{8f_1 + 2f_2}{720} \quad (5.6)$$

$$\hat{E} \equiv \frac{8f_1 + 2f_2}{2160} \quad (5.7)$$

$$\hat{F} \equiv (8f_1 + 2f_2)\left[\frac{\gamma}{720} - \frac{1}{720}\ln(4\pi) - \frac{683}{129600}\right] . \quad (5.8)$$

Applying the same techniques to the next highest order pieces leads to

$$\Pi(q^2)_{SE} = \frac{g^2}{(4\pi)^8} q^8 \left[\frac{A'}{\epsilon^2} + B' \ln\left(-\frac{q^2}{\nu^2}\right) + C' \ln^2\left(-\frac{q^2}{\nu^2}\right) + \frac{D'}{\epsilon} \ln\left(-\frac{q^2}{\nu^2}\right) + \frac{E'}{\epsilon} + F' \right] \quad (5.9)$$

and

$$\Pi(q^2)_{GE} = -\frac{g^2}{(4\pi)^8} q^8 \left[\frac{\tilde{A}}{\epsilon^2} + \tilde{B} \ln\left(-\frac{q^2}{\nu^2}\right) + \tilde{C} \ln^2\left(-\frac{q^2}{\nu^2}\right) + \frac{\tilde{D}}{\epsilon} \ln\left(-\frac{q^2}{\nu^2}\right) + \frac{\tilde{E}}{\epsilon} + \tilde{F} \right], \quad (5.10)$$

where

$$A' \equiv \frac{C_2(R)(8f_1 - f_2)}{960} \quad (5.11)$$

$$\tilde{A} \equiv \frac{f_6}{720} \quad (5.12)$$

$$B' \equiv C_2(R) \left[8f_1 \left(\frac{-21}{320} + \frac{\gamma}{60} - \frac{1}{60}\ln(4\pi) \right) + 2f_2 \left(\frac{377}{5760} - \frac{\gamma}{120} - \frac{1}{120}\ln(4\pi) \right) \right] \quad (5.13)$$

$$\tilde{B} \equiv f_6 \left[-\frac{67}{720} + \frac{\gamma}{45} + \frac{1}{45}\ln(4\pi) \right] \quad (5.14)$$

$$C' \equiv \frac{C_2(R)(8f_1 + f_2)}{120} \quad (5.15)$$

$$\tilde{C} \equiv \frac{f_6}{90} \quad (5.16)$$

$$D' \equiv \frac{C_2(R)(8f_1 + f_2)}{240} \quad (5.17)$$

$$\tilde{D} \equiv \frac{f_6}{180} \quad (5.18)$$

$$E' \equiv C_2(R) \left[8f_1 \left(-\frac{21}{1280} + \frac{\gamma}{240} - \frac{1}{240}\ln(4\pi) \right) - 2f_2 \left(\frac{377}{23040} - \frac{\gamma}{480} + \frac{1}{480}\ln(4\pi) \right) \right] \quad (5.19)$$

$$\tilde{E} \equiv \frac{f_6}{2} \left[-\frac{67}{1440} + \frac{\gamma}{90} - \frac{1}{90}\ln(4\pi) \right] \quad (5.20)$$

$$F' \equiv C_2(R) \left[8f_1 \left(-\frac{\pi^2}{2880} + \frac{\gamma^2}{120} - \frac{21\gamma}{320} + \frac{21847}{138240} + \frac{21}{320}\ln(4\pi) - \frac{\gamma}{60}\ln(4\pi) + \frac{1}{120}\ln^2(4\pi) \right) - 2f_2 \left(-\frac{13967}{55296} + \frac{\pi^2}{5760} - \frac{\gamma^2}{240} + \frac{377\gamma}{5760} - \frac{377}{5760}\ln(4\pi) + \frac{\gamma}{120} - \frac{1}{240}\ln^2(4\pi) \right) \right] \quad (5.21)$$

$$\tilde{F} \equiv f_6 \left[-\frac{67\gamma}{720} - \frac{\pi^2}{2160} + \frac{\gamma^2}{90} + \frac{24169}{103680} + \frac{67}{720}\ln(4\pi) - \frac{\gamma}{45}\ln(4\pi) + \frac{1}{90}\ln^2(4\pi) \right]. \quad (5.22)$$

Combining all of these terms together we obtain the final result for the lowest and second order perturbative pieces of the correlation function

$$\begin{aligned}\Pi(q^2) =& \frac{(q^2)^4}{(4\pi)^6}\left[\hat{B}\ln\left(-\frac{q^2}{\nu^2}\right)+\frac{\hat{E}}{\epsilon}+\hat{F}\right]+\frac{g^2(q^2)^4}{(4\pi)^8}\left[\frac{A}{\epsilon^2}+B\ln\left(-\frac{q^2}{\nu^2}\right)\right.\\ &\left.+C\ln^2\left(-\frac{q^2}{\nu^2}\right)+\frac{D}{\epsilon}\ln\left(-\frac{q^2}{\nu^2}\right)+\frac{E}{\epsilon}+F\right],\end{aligned} \tag{5.23}$$

where the coefficients A,B,C,D,E and F are given by

$$A \equiv 4A' - 2\tilde{A} \tag{5.24}$$

$$B \equiv 4B' - 2\tilde{B} \tag{5.25}$$

$$C \equiv 4C' - 2\tilde{C} \tag{5.26}$$

$$D \equiv 4D' - 2\tilde{D} \tag{5.27}$$

$$E \equiv 4E' - 2\tilde{E} \tag{5.28}$$

$$F \equiv 4F' - 2\tilde{F}\,. \tag{5.29}$$

5.1 Summary

The process of obtaining a mass of a particular state using QCD sum rules involves many steps. The calculation, in the chiral limit ($m_q \to 0$), of the perturbative piece of the correlation function $\Pi(q^2)$ for a $q\bar{q}q\bar{q}$ state to second order performed in this thesis is a crucial step in determining the mass of this state. Although not all of the second order terms have been determined, and a final sum-rule prediction for the mass of our $q\bar{q}q\bar{q}$ state remains elusive, there are still several things that can be deduced from (5.23). First, all the colour and flavour information for this state are carried in the f_i's ($i = 1, 2, 3, 4, 6$). The $\frac{1}{\epsilon}\ln\left(-\frac{q^2}{\nu^2}\right)$ term implies that there has been some operator mixing and that operator renormalization is needed. As predicted, the renormalization of the composite operators in our current $J(x)$ will affect the calculation of $\Pi(q^2)$ and cannot be ignored.

Further work will be needed to obtain a mass for this state using QCD sum rules. In order to find the missing second order pieces of $\Pi(q^2)$ we will need to take the operator mixing into account. Once the missing terms have been discovered they should cancel out the $\frac{1}{\epsilon}\ln\left(-\frac{q^2}{\nu^2}\right)$ term and the only divergences remaining can be eliminated using a proper dispersion relationship.

Once a dispersion relationship has been applied to the correlation function we can determine if the second order terms in the perturbative expansion of $\Pi(q^2)$ play a large role, as they do for $q\bar{q}$ scalar mesons [16], in improving the accuracy in the calculation of the mass of our $q\bar{q}q\bar{q}$ state. From there we could examine the masses of several of the four-quark candidates from the scalar mesons to see if there is a match. Such a conclusion would be a welcome discovery for the Standard model since it would be one step closer to solving the scalar meson problem.

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Appendix A

Appendix (Conventions and Notation)

A.1 Units

$$\hbar = c = 1 \longrightarrow 1 \approx 200\text{MeV} \cdot \text{fm} \tag{A.1}$$

Fine-structure constant at low energy is

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}\,. \tag{A.2}$$

Strong coupling constant is

$$\alpha_s = \frac{g^2}{4\pi}\ , \quad \alpha_s(M_Z) \approx 0.119\,. \tag{A.3}$$

A.2 Special Relativity

The 4-vector p^μ is

$$p^\mu = \left(p^0, \vec{p}\right) \tag{A.4}$$

The metric $g^{\mu\nu}$ is

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\,. \tag{A.5}$$

The dot product is

$$p \cdot q = g^{\mu\nu} p_\mu q_\nu = g_{\mu\nu} p^\mu q^\nu = p^\mu q_\mu = p_\mu q^\mu\,. \tag{A.6}$$

A.3 γ (Dirac) Matrices

The following are the Pauli spin matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{A.7}$$

The Dirac matrices are

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\,. \tag{A.8}$$

The γ_5 matrix satisfies the following relations [17, 34]:

$$(\gamma_5)^\dagger = \gamma_5\,; \tag{A.9}$$

$$(\gamma_5)^2 = 1\,; \tag{A.10}$$

$$\{\gamma_5, \gamma_\mu\} = 0\,. \tag{A.11}$$

In four dimensions, the explicit definition of γ_5 is

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \,. \tag{A.12}$$

The γ matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{A.13}$$

in D dimensions

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} - (4-D)\gamma^\nu\gamma^\rho \,. \tag{A.14}$$

A.4 Trace Theorems

The notation

$$\not{a} \equiv \gamma^\mu a_\mu \tag{A.15}$$

will be used. The following are the D-dimensional trace theorems that are used in the calculations in this thesis:

$$Tr[\gamma_5 \not{a}\gamma_5 \not{b}] = 4a \cdot b \,, \tag{A.16}$$

$$Tr[\gamma_5 \not{a}\gamma_5 \not{b}\gamma_5 \not{c}\gamma_5 \not{d}] = 4[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)] \,, \tag{A.17}$$

$$\begin{aligned} Tr[\gamma_5 \not{a}\gamma_5 \not{b}\gamma^\mu \not{c}\gamma_5 \not{d}\gamma_\mu] =16 \Bigg[&(a\cdot b)(c\cdot d) - \frac{(4-D)}{4}[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) \\ &+ (a\cdot d)(b\cdot c)] \Bigg] \,. \end{aligned} \tag{A.18}$$

A.5 Noether's Theorem

Noether's theorem covers the connection between symmetries and conservation laws in field theory. We are concerned with infinitesimal, continuous transformations on the fields ϕ given as

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi(x) \,, \tag{A.19}$$

where α is the infinitesimal parameter and $\Delta\phi$ is a deformation of the field. If the action ($S = \int Ldt$) is invariant under the above transformation then the transformation is a symmetry. This implies that the Lagrangian must be invariant under the transformation up to a 4-divergence:

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + \alpha\partial_\mu\mathcal{J}^\mu(x) \tag{A.20}$$

for some $\mathcal{J}^\mu$. Under a variation of the fields, we find

$$\Delta\mathcal{L} = \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\Delta\phi \,. \tag{A.21}$$

The Euler-Lagrange equation tells us that the second term in the above equation vanishes. Set the first term equal to $\alpha\partial_\mu\mathcal{J}^\mu$ and we obtain the result of Noether's theorem:

$$\partial_\mu j^\mu(x) = 0 \tag{A.22}$$

for

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu \,. \tag{A.23}$$

This implies that the current $j^\mu(x)$ is conserved.

A.6 Gamma $\Gamma(z)$ and Beta $B(z,w)$ functions

When calculating standard Feynman integrals we will obtain answers in the form of Gamma Γ functions. The definition of the Γ function is

$$\Gamma(z) \equiv \int_0^\infty dt\ t^{z-1} e^{-t}\,, \tag{A.24}$$

with

$$\Gamma(1) = 1\,. \tag{A.25}$$

It is also useful to know the recursion relation

$$\Gamma(z+1) = z\Gamma(z)\,. \tag{A.26}$$

The Gamma function $\Gamma(z)$ can be expanded about $z=0$ as

$$\Gamma(z) = \frac{1}{z} - \gamma + \left(\frac{\pi^2}{12} + \frac{\gamma^2}{2}\right) z + \left(-\frac{1}{3}\zeta(3) - \frac{\pi^2\gamma}{12} - \frac{\gamma^3}{6}\right) z^2 + O(z^3)\,, \tag{A.27}$$

where $\gamma \approx 0.577216$ is the Euler-Mascheroni constant and $\zeta(x)$ is the zeta function.

Another important function is the Beta function which is defined as

$$B(z,w) \equiv \int_0^1 dt\ t^{z-1}(1-t)^{w-1} \qquad , Re\ z > 0, \quad Re\ w > 0\,, \tag{A.28}$$

which can be related to the $\Gamma(z)$ function

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}\,. \tag{A.29}$$

A.7 Values for Two-Loop Feynman Integrals

The following are the values of several two-loop Feynman integrals that are used in the derivation of (3.76):

$$\begin{aligned} I_2'(2,1,1,1,0) &= \frac{1}{(\nu)^{4\epsilon}} \int\int \frac{d^D k_1 d^d k_2}{(2\pi)^{2D}} \frac{1}{(k_1^2)^2 k_2^2 (k_1+q)^2 (k_2+q)^2} \\ &= \frac{i^2}{(4\pi)^4}\left(-\frac{1}{4\pi\nu^2}\right) \frac{\Gamma(1+\epsilon)^3\Gamma(-\epsilon)\Gamma(\epsilon)\Gamma(1-\epsilon)}{\Gamma(2+2\epsilon)\Gamma(1+2\epsilon)} (q^2)^{2\epsilon-1} \end{aligned} \tag{A.30}$$

$$\begin{aligned} I_2'(2,0,1,1,1) &= \frac{1}{(\nu)^{4\epsilon}} \int\int \frac{d^D k_1 d^d k_2}{(2\pi)^{2D}} \frac{1}{(k_1^2)^2 (k_1+q)^2 (k_2+q)^2 (k_2-k_1)^2} \\ &= \frac{i^2}{(4\pi)^4}\left(-\frac{1}{4\pi\nu^2}\right) \frac{\Gamma(1+\epsilon)^2\Gamma(-\epsilon)\Gamma(\epsilon)\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)\Gamma(1+3\epsilon)\Gamma(2+2\epsilon)} (q^2)^{2\epsilon-1} \end{aligned} \tag{A.31}$$

$$\begin{aligned} I'(1,1,2,1,0) &= \frac{1}{(\nu)^{4\epsilon}} \int\int \frac{d^D k_1 d^d k_2}{(2\pi)^{2D}} \frac{1}{k_1^2 k_2^2 [(k_1+q)^2]^2 (k_2+q)^2} \\ &= I'(2,0,1,1,1) \end{aligned} \tag{A.32}$$

$$\begin{aligned} I'(1,1,2,0,1) &= \frac{1}{(\nu)^{4\epsilon}} \int\int \frac{d^D k_1 d^d k_2}{(2\pi)^{2D}} \frac{1}{k_1^2 k_2^2 [(k_1+q)^2]^2 (k_2-k_1)^2} \\ &= I'(2,0,1,1,1) \end{aligned} \tag{A.33}$$